Dynamics of a Spin Stabilized Spacecraft Having Flexible Appendages(Dissertation_全文)

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ABSTRACT

This dissertation treats analytically dynamics of a spin stabilized spacecraft having flexible appendages. The spacecraft is modeled as a central rigid body having attached to it flexible appendages lying in a plane which contains the center of mass and is normal to the spin axis.

1. The attitude stability of the spacecraft in a force free environment is investigated using the Liapunov direct method. The total energy of the system, constrained through the angular momentum integral, is used as a Liapunov function. The approach to the stability problem is based on the functional analysis. Necessary and sufficient conditions for the stability of the attitude motion are established for this spacecraft model. Sufficient conditions are also obtained in simple forms.

2. On the basis of linearized equations of motion, the attitude behavior of this class of spacecraft in a force free environment is investigated using an analytical method which utilizes the method of averaging. The damping ratio and the frequency of nutational body motions are obtained. Attitude stability criteria are also deduced from the sign properties of the damping ratio. These results are compared with numerical solutions.

3. Analysis is made of a heavy damping of nutational body motions of this class of spacecraft due to a certain nonlinear internal resonance between vibrations of the appendages and nutational body motions. The method of averaging is used to derive an analytical expression for the damping of nutational body motions. The accuracy of this expression is confirmed by digital computer simulations.

4. It is shown that this class of spacecraft may exhibit a steady nutational

body motion induced by solar heating: Thermally induced vibrations of the appendages cause a periodic variation of the moments of inertia of the spacecraft and this in turn, through the parametric excitation, produce a self-excitation of nutational body motions. The amplitude of the nutational body motion is determined by means of the method of averaging and the stability of the motion is discussed in detail.

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CHAPTER I

INTRODUCTION

1.1 Scope

The motion of a spacecraft can be described in many cases by the translational motion of the mass center and the rotational motion of a spacecraft about its center of mass. The latter motion is refered to as the attitude motion and forms the object of our interest.

Because of functional requirements, a spacecraft is required to maintain specified orientations with respect to an inertia space. The spacecraft attitude may be controlled by active or passive means. For spacecraft missions of long duration, the use of passive techniques appears attractive, especially when pointing accuracy requirements are not stringent. One simple means of maintaining the spacecraft attitude in space is to provide a spin about the axis which is to be controlled (Spin Stabilization). This dissertation will focus attention upon the motions of spin stabilized spacecrafts.

A space flight involves two distinct and radically different dynamic environments : A brief interval of vigorous accelerations and vibrations during boost, followed by prolonged functioning in a quiescent mode under extremely small loads and accelerations, characterizes every spacecraft flight. In order to circumvent the dilemma this poses, spacecraft designers have usually adopted lightweight (and extremely flexible) deployable appendages. The resulting vehicle is relatively compact and rigid during the launch phase of its history, and after boost termination, the appendages emerge from the rigid core until the structure undergoes complete metamorphosis. Although the design of large flexible spacecraft structures can be made quite adequate in terms of structural loads for the nominal in-orbit environment, it is possible that the compliance of such limber structures can interact unfavorably with the spacecraft attitude motion. Hence, an analysis of the effect of flexible appendages on the attitude motion of the spacecraft becomes an important subject in the study of the dynamics of spacecraft. The effect of flexible appendages on the attitude motion of a spin stabilized spacecraft will form the central theme of the present dissertation.

The influence of flexible appendages on the attitude motion of a spin stabilized spacecraft was first observed in the Explorer I. The Explorer I was to be passively spin stabilized about an axis of minimum moment of inertia. After only one orbit, however, the spacecraft exhibited a nutational body motion. Within a few days, the spacecraft achieved a steady spinning motion about an axis of maximum moment of inertia. In an attempt to explain the nutational body motion of the Explorer I, Bracewell and Garriot(1) and Thomson and Reiter (2) have investigated the effect of the energy dissipation resulting from vibrations of certain elastic parts of a spacecraft. In these investigations, a simple approximation technique has been employed : It is to assume that the moments of inertia of a spacecraft do not vary significantly and the angular momentum of the relative motion within a spacecraft is negligible compared to the rigid body motion. Then, the relative motion is idealized as a slow removal of energy (energy sink) and the rate of the energy dissipation can be related to the change in the attitude motion of the spacecraft. With this relationship, an estimate of a nutational body motion of the spacecraft is easily obtained. Stability conditions of the attitude motion are also obtained from the sign properties of the damping ratio for the nutational body motion. Bracewell and Garriot[1] and Thomson and Reiter[2], using this technique, have shown that with a small amount of energy disipation a freely spinning body is stable only about an axis of maximum moment of inertia (the maximum inertia axis criterion). Since then, because of its simplicity, this approximation technique, known as the energy sink method or the "quasi rigid body model", has been widely used as an analytical tool for qualitative determination of the influence of energy dissipation on the attitude motion of a spin stabilized spacecraft.

As long as a spacecraft is relatively rigid, the results obtained from the energy sink method are valid. Modern spacecrafts are, however, far from rigid : Current designs of spacecraft employ large, highly flexible appendages as antennas or solar arrays to meet increasingly demanding missions. The presence of large flexible appendages on a spacecraft necessitates re-examining the basic characteristics of the dynamics of the spacecraft. There are three important subjects in the study of the dynamics of a spin stabilized spacecraft having flexible appendages of dynamically significant importance;

(1) To establish stability conditions of the attitude motion of this class of spacecraft.

(2) To develop a mathematical procedure for qualitative determination of the influence of elastic vibrations of the appendages on the attitude motion of this class of spacecraft.

(3) To predict the anomalous behavior of this class of spacecraft due to dynamic interactions of the flexible appendages with the environments and with the attitude motion of the spacecraft.

There is some information in the literature on the attitude stability of a freely spinning spacecraft having flexible appendages. Buckens [6,7] has investigated the effect of flexible appendages on the stability of motion of a freely spinning spacecraft by analytical means and given thresholds of instability regions by the values of spin velocities for which nutational frequency as measured in the spacecraft first becomes zero. His works, while being the earliest attempt to derive the stability criteria for this class of spacecraft, are heuristic and preliminary in character because of the absence of mathematical rigor. Meirovitch and Nelson [8] have presented an analysis of a freely spinning spacecraft with flexible antennas extending along the spin axis. The quality of the spinning motion was investigated for various values of the system parameters by means of digital computer eigenvalue analysis; it was concluded that the attitude stability of this class of spacecraft depends on the magnitude of the ratio of the natural frequency of the antennas to the spin velocity. Vigneron [9] has studied the dynamics of a spin stabilized spacecraft with long crossed dispoles and derived necessary and sufficient conditions for stability of the spin axis orientation with the aid of a digital computer using Routh's rules. The results have been expressed in the form of a parameter chart. Recently, Pringle [10,11,12] has discussed the stability of damped mechanical systems by using the Liapunov direct method and shown that the Hamiltonian of the system is a very useful testing function for asymptotic stability and instability. Since then, the Liapunov direct method has been widely used to analyze the stability of a freely spinning spacecraft having flexible appendages and the Hamiltonian of the system has been used as a Liapunov function. These analyses have yielded closed form conditions for stability of the attitude motion for restricted spacecraft models. Meirovitch[13] and Meirovitch and Calico[14] have delt with a freely spinning spacecraft having flexible appendages directed along the spin axis and obtained sufficient conditions for stability. Hughes and Fung [15] have provided sufficient conditions for stability for a spinning spacecraft having flexible appendages normal to the spin axis rotating freely in space. However, the Hamiltonian, employed in the analysis, did not take into account the existence of the angular momentum integral. Taking into consideration the angular momentum integral, Barbera and Likins [16] have delt with essentially the same system studied by Hughes and Fung [15] and obtained the necessary and sufficient conditions for stability of the attitude motion for this class of spacecraft. The advantage of the Liapunov direct method is shown by the fact that it permits the derivation of closed form stability criteria. However, the approaches used in the analyses involve certain discretization procedures, so that the question remains as to the effect of the discretization processes on the results. Hence, it is highly desirable to develop a new approach which avoids a discretization process and permits a more rigorous analysis.

There has been very little attention given to the second subject of the study. Until recently, no attempts have been made to extend the energy sink method to a spinning spacecraft having large flexible appendages : There has existed no analytical approach which can be suitably used in the treatment of the attitude motion of this class of spacecraft. Hence, it becomes important and desirable to develop a new analytical method which will be sufficiently accurate to provide the influence of elastic vibrations of the appendages on the attitude motion of this class of spacecraft.

For a spacecraft with large flexible components, it is possible that these flexible components interact unfavorably with the attitude motion of the spacecraft. Recent flight experience has shown much anomalous behavior due to complex interactions of flexible appendages with the attitude motion of the spacecraft [17] For example, the Alouette I, II and the Explorer XX exhibited rapid spin decays due to solar radiation pressure on flexible appendages deformed by solar heating. The OGO IV began to oscillate after deployment of long flexible appendages. This phenomenon has been explained as the consequence of interactions between thermally induced vibrations of the appendages and attitude motions of the spacecraft. Several investigations on the flight anomalies of spinning flexible spacecrafts have been described in the literature. In an attempt to explain the spin decay phenomena of the Allouette I, II and the Explorer XX, Etkin and Hughes [18] have investigated the despin mechanism of a spinning spacecraft with flexible appendages due to the action of solar radiation. Vigneron and Boresi [19] have presented the analysis of the long term spin decay caused by the structural damping combined with the gravity field. The attitude dynamics of a spinning spacecraft during extension of flexible apepndages has been studied analytically and numerically by Canadian scientists. [20,21,22] On the other hand, Pringle [23] has discussed the dynamics of a spacecraft composed of a pair of point masses connected by a spring (Dumbbell Satellite) and demonstrated that this class of spacecraft can exhibit a heavy damping of attitude motions due to a certain nonlinear internal resonance. Others who have considered the anomalous behavior of a gravity gradient dumbbell satellite are Austin [24], Tai and Loh [25], Chobotov [26] and Crist and Eisley [27], but their works are somewhat peripheral to the subject at hand. In an attempt to explain the oscillation of the OGO IV, thermally induced vibrations of appendages has been studies by a number of authors [28,29,30], but the works have been restricted only to appendage motions and not concerned with the attitude motion of the spacecraft. Dynamic behavior of a spacecraft having flexible appendages is usually considered during the design of a spacecraft structure and the possibility of severe interactions of flexible appendages with the attitude motion of the spcaecraft is usually considered. Less severe interactions may go undetected, however, but may still be of such magnitude as to result in a possible failure of the spacecraft to complete all or part of its mission. From experience gained in the past, it is clear that such interactions usually result from a lack of knowledge about the mechanism causing the interactions. Special emphasis should, therefore, be placed on investigating possible interactions of flexible appendages with the attitude motion of the spacecraft in sufficient depth and detail.

1.2 Contributions

This dissertation is devoted to analytical studies of the dynamics of a spin stabilized spacecraft having flexible appendages. The spacecraft is conceived as a central rigid body having attached to it flexible appendages lying in a plane which contains the center of mass and is normal to the spin axis. The major subjects of this study are :

(1) To establish necessary and sufficient conditions for stability of the attitude motion for this class of spacecraft in a force free environment.

(2) To develop an analytical method which is suitable for exploring the basic characteristics of the dynamics of this class of spacecraft.

(3) To predict the anomalous behavior of this class of spacecraft which is attributable to the dynamic interactions of the flexible appendages with the environments and with the attitude motions of the spacecraft.

Chapter II deals with the attitude stability of this class of spacecraft in a force free environment on the basis of the Liapunov direct method. The total energy of the system, which is constrained through the angular momentum integral, is used as a Liapunov function. The approach to the stability problem is based on the functional analysis : The restricted total energy of the system is described in terms of the angular velocity of the spacecraft and elastic displacement of the appendages. Stability of the motion is determined from the sign properties of a certain functional depending on elastic displacements of the appendages alone. This approach permits a rigorous analysis which avoids the questions as to the effect of discretization processes on the results. Necessary and sufficient conditions for the stability of the attitude motion for this spacecraft model are established and sufficient conditions are also obtained in simple forms.

Chapter III is concerned with the development of an analytical procedure which is suitable for exploring the dynamic characteristics of this class of spacecraft. First, linearized equations of motion for a freely spinning spacecraft with flexible appendages are formulated. The hybrid coordinate method in conjunction with series truncation is employed : Motions are described in terms of the angular velocity of the spacecraft and modal deformation coordinates of the appendages. Then, the equations are solved by an analytical method which utilizes the method of averaging, and closed form approximate solutions are obtained. The damping ratio and the frequency of nutational body motions are derived from the solutions. Attitude stability criteria are also obtained from examination of the sign properties of the damping ratio.

The last two chapters are concerned with the analytical studies of the anomalous behavior of this class of spacecraft which is attributable to the dynamic interactions of the flexible appendages with the attitude motions of the spacecraft and with the enviornments. The method of averaging is used as an analytical tool.

Chapter IV deals with the heavy damping characteristics of nutational body motions due to a certain nonlinear internal resonance : When the natural frequencies of the appendages are nearly equal to twice the nutational frequency, a large energy transfer takes place between vibrations of the appendages and nutational body motions and the disipation of energy derived from it results in a heavy damping of nutational body motions. An analytical expression for the damping of nutational body motions is obtained.

Chapter V demonstrates the thermally induced nutational body motions of a spinning spacecraft with flexible appendages : When this class of spacecraft is exposed to solar radiation, thermally induced vibrations of the appendages occur at a spin frequency, which causes a periodic variation of the moments of inertia at that frequency and this in turn, through the parametric excitation, produces a self-excitation of nutational body motions. The amplitude of the nutational body motion is determined analytically and stability of the motion is examined in detail.

An appendix is annexed to this dissertation. The appendix presents the kinetic energy and angular momentum expressions of a spacecraft having flexible appen appendages.

CHAPTER II

LIAPUNOV STABILITY ANALYSIS OF A FLEELY SPINNING SPACECRAFT HAVING FLEXIBLE APPENDAGES

2.1 Introduction

Knowledge of the attitude stability of a spacecraft is of fundamental importance in the basic design of a spacecraft structure. It is well known that the rotational motion of a torque free rigid body is stable if the rotation takes place about an axis corresponding to either maximum or minimum moments of inertia but the motion is unstable if the rotation takes place about an axis of intermediate moment of inertia. In general, however, spacecrafts are not entirely rigid and the question remains as to what extent the rigid body idealization can be utilized in analyzing a nonrigid spacecraft.

On the influence of vehicle nonrigidity on the spacecraft attitude stability, there is much information in the literature, but for the most part, it has treated the special case of a quasirigid body model: The attitude motion of a spacecraft has been discussed, on the assumption that the relative motion within the spacecraft is considered to remove mechanical energy without being coupled with the motion of the spacecraft. The attitude stability of a freely spinning spacecraft with flexible appendages of immediate dynamic significance has received considerable attention in recent years [7-16] The Liapunov direct method in particular has been successfully applied to the stability problem of this class of spacecraft. [13-16] The Hamiltonian of the system has been used as a Liapunov function. These analyses have yielded closed form stability criteria. However, the approaches used involve certain discretization procedures, so that the uncertainty remains concerning the effect of the discretization procedures on the results obtained.

The purpose of this chapter is to generate necessary and sufficient conditions for the attitude stability of a freely spinning spacecraft having flexible appendages on the basis of the Liapunov direct method. The total energy of the system, constrained through the angular momentum integral, is used as a Liapunov function. The approach to the stability problem is based on the functional analysis : The restricted total energy of the system is described in terms of the angular velocity of the spacecraft and elastic displacements of the appendages. The stability for the motion is determined from the sign properties of a certain functional depending on elastic displacements of the appendages alone. This approach permits a rigorous analysis which avoids the questions as to the effect of discretization processes on the results. The spacecraft is modelded as a rigid body having attached to it long flexible appendages lying in a plane which contains the center of mass and is normal to the spin axis (Fig. 2.1)

Necessary and sufficient conditions for the stability of the attitude motion are determined for this spacecraft model in a force free environment. Furthermore, sufficient conditions are also obtained in simple forms.

2.2 Definitions and Basic Theorems

We, here, give certain statements about a freely rotating system and theorems on the stability of the system. The system is idealized as a main body with holonomically constrained moving parts. The constraints do not contain the time t. In this section, to simplify the results, we shall assume that the appendages are idealized as interconnected particles; all the formulae can be at once generalized to the case of distributed ones. [33]

In the construction of the equations of motion of such mechanical systems, it is effective to use the angular velocity of the main body and generalized coordinates for the moving parts. The Lagrange equations of motion of the system are written in terms of these variables as follows : [32]

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + \omega_2\left(\frac{\partial T}{\partial \omega_3}\right) - \omega_3\left(\frac{\partial T}{\partial \omega_2}\right) = 0\\
&\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) + \omega_3\left(\frac{\partial T}{\partial \omega_1}\right) - \omega_1\left(\frac{\partial T}{\partial \omega_3}\right) = 0\\
&\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) + \omega_1\left(\frac{\partial T}{\partial \omega_2}\right) - \omega_2\left(\frac{\partial T}{\partial \omega_1}\right) = 0\\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = Q_i \qquad (i=1, \ldots\ldots, N)
\end{aligned}\right\} \tag{2.1}$$

where T is the kinetic energy of the system, L the Lagrangian of the system, ω_1, ω_2, ω_3 the angular velocity components about the orthonormal axes X_1, X_2, X_3, respectively (the axes X_1, X_2, X_3 are fixed in the main body.), q_i the generalized coordinates for the moving parts, Q_i the generalized forces. The Lagranigian L is defined by

$$L = T - U$$

where U is the potential energy of the system. The generalized forces Q_i are defined in terms of any nonconservative applied forces. One possible motion for the system is that in which the spin axis remains inertially at rest and the moving parts assume equilibrium positions. By a proper selection of the coordinates, this motion corresponds to the solution

$$\omega_1 = 0, \qquad \omega_2 = 0,$$

$$\omega_3 = \omega_o \text{ (a constant)}, \quad q_i = 0 \; (1=1, \ldots\ldots, N) \Bigg] \qquad (2.2)$$

Since we consider the system in which the constraints do not involve the time t, the total energy H of the system is given by [32]

$$H = \sum_{i=1}^{3} \omega_i \frac{\partial T}{\partial \omega_i} + \sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L. \qquad (2.3)$$

It is important to calculate the time derivative of the total energy H of the system. This may be shown to be (by the use of Eqs. (2.1))

$$\frac{dH}{dt} = \sum_{i=1}^{N} Q_i \dot{q}_i \,. \qquad (2.4)$$

The term $\sum_{i=1}^{N} Q_i \dot{q}_i$ is the power into the system by the generalized forces Q_i . Hence, it follows that H decreases monotonically with time for the system in which the power is negative. Such reductions in H are usually associated with the energy dissipation in a physical system. When it can be established that the total energy is strictly diminishing with time for any motion in the neighborhood of the nominal motion to be examined for stability, then this function becomes extremely valuable in the stability analysis by the Liapunov direct method. A notion which is useful in investigation of the behavior of the total energy H is that of "pervasive damping". [34]

Definition : A mechanical system with a pervasive damping is one in which the power is strictly negative for any path which is neighboring but not identical to the path corresponding to the nominal motion.

The system considered here is free of external forces, so that the three torque components about the mass center of the system are zero. It follows that the angular momentum vector $\underline{L}$ $(\omega_1, \omega_2, \omega_3, \dot{q}_i, q_i)$ about the mass center is conserved : It represents a motion integral. Let it be expressed as follows :

$$\ell^2 = \sum_{i=1}^{3} L_i^2(\omega_1, \omega_2, \omega_3, \dot{q}_i, q_i) \tag{2.5}$$

Where ℓ is the magnitude of the angular momentum vector of the system, L_i the angular momentum components along the X_1 axes. Equation (2.5) can be used to eliminate one of the angular velocity components : Solving for ω_3 from Eq. (2.5) and substituting into Eq. (2.3), we get

$$H = H(\omega_1, \omega_2, \dot{q}_i, q_i \,;\, \ell^2). \tag{2.6}$$

Then, the nominal motion (2.2) becomes the origin of the phase space formed by $\omega_1, \omega_2, \dot{q}_i, q_i$ with a phase vector $\underset{\sim}{x} = (\omega_1, \omega_2, \dot{q}_i, q_i)$ describing the departure from the nominal motion. Without loss of generality, it can be assumed that the function H vanishes at the origin of the phase space.

The word "stable", when applied in the astrospace field to freely spinning vehicles, is to mean that a vehicle initially freely rotating with any axis fixed in the main body will return to that state if subjected to any sufficiently small perturbation. Without losing this sense of the word, we adopt a set of more precise analytical definitions.

Definition : If for any $\epsilon > 0$, there exists $\delta > 0$ such that the initial perturbation restriction $|\underset{\sim}{x}_o| \leqq \delta$ implies, for all $t > t_o$, $|\underset{\sim}{x}(t; t_o, \underset{\sim}{x}_o)| \leqq \epsilon$, the solution (2.2) is Liapunov stable.

Definition : If the solution (2.2) is Liapunov stable and furthermore the limit of $\underset{\sim}{x}$ vanishes as the time t approaches infinity, the solution (2.2) is asymptotically stable.

Definition : The term "unstable" means not Liapunov stable.

In application of the Liapunov direct method to freely rotating systems, we can make use of the total energy $H(\underline{x})$ of the system, constrained through the angular momentum integral, as a Liapunov function. Then the following theorem can be stated : (10,12)

Theorem : A freely rotating system with a pervasive damping is asymptotically stable if $H(\underline{x})$ is a positive definite function of $\underline{x}$, and 2) unstable if $H(\underline{x})$ can take on negative values for $\underline{x}$ arbitrarily close to $\underline{x} = 0$.

Part 1 of the theorem is proved using the asymptotic stability theorem in Ref. 35, P. 37 with $H(\underline{x})$ as a Liapunov function. Part 2 is proved using the first instability theorem in Ref. 35, P. 38 with $H(\underline{x})$ as a testing function. The important quality of this theorem is the fact that it offers conditions both necessary and sufficient for the asymptotic stability for a freely rotating system with a pervasive damping. The usual limitation on the application of the Liapunov direct method is that the stability theorem provides only sufficient conditions.

2.3 Energy and Angular momentum expressions

Let us consider a spinning spacecraft composed of a central rigid body B and long flexible appendages A_i as shown in Fig. 2.1.

The appendages lie in a plane containing the mass center and is normal to the spin axis, when the vehicle is steadily spinning. It is assumed that the appendages are built in at the bases, free at the upper ends : No displacement is possible, and moreover no bending is possible at the bases.

A set of body axes X_1, X_2 and X_3 (unit vectors are $\underline{b}_1$, $\underline{b}_2$ and $\underline{b}_3$) are fixed along the principal axes of the system in the nominally undeformed configuration. The X_3 axis coincides with the spin axis. For an appendage i, a set of axes ζ_i,

η_i, and ξ_i (unit vectors are $\underline{a}_{i1}$, $\underline{a}_{i2}$ and $\underline{a}_{i3}$) is defined so that the ζ_i axis coincides with the apepndage i in the undeformed state and the ξ_i axis coincides with the spin axis. The axes X_1 and ζ_i make an angle γ_i. The two sets of axes are related by the direction cosine matrix C_i, i.e., the column arrays of vectors $[\underline{b}] = [\underline{b}_1, \underline{b}_2, \underline{b}_3]^T$ and $[\underline{a}_i] = [\underline{a}_{i1}, \underline{a}_{i2}, \underline{a}_{i3}]^T$ are related by

$$\left.\begin{array}{l} [\underline{a}_i] = C_i\,[\underline{b}] \\ C_i = \begin{bmatrix} C\gamma_i & S\gamma_i & 0 \\ -S\gamma_i & C\gamma_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}\right\} \tag{2.7}$$

where the superscript T denotes the transpose of a matrix or a vector array and $C\gamma_i = \cos\gamma_i$, $S\gamma_i = \sin\gamma_i$.

Let P be the mass center of the total configuration and let O be coincident with P when the vehicle is undeformed (Fig. 2.1). Vector $\underline{c}$ is the vector from P to the point O. Let us denote $\underline{\rho}$ a vector from O to an element of mass dm in B. The position of a typical point Q_i in an undeformed appendage i relative to the point O is denoted by a vector $\underline{r}_i$ and an elastic deformation of the appendage i at the point Q_i by $\underline{w}_i$.

The system kinetic energy T is given by

$$2T = \int_B (\dot{\underline{c}} + \dot{\underline{\rho}}) \cdot (\dot{\underline{c}} + \dot{\underline{\rho}})dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} (\dot{\underline{c}} + \dot{\underline{r}}_i + \dot{\underline{w}}_i) \cdot$$

$$(\dot{\underline{c}} + \dot{\underline{r}}_i + \dot{\underline{w}}_i)\, ds_i \tag{2.8}$$

where $\int_B dm$ denotes that the integration is carried out over the body B, μ_i and ds_i the mass per unit length and the arc length along an appendage i, respectively, ℓ_i the length of an appendage i. The dot over a vector denotes the time differentiation of that vector with respect to an inertia frame.

The mass center definition requires that

$$\int_B (\underline{c} + \underline{\rho})dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} (\underline{c} + \underline{r}_i + \underline{w}_i)\, ds_i = 0 . \tag{2.9}$$

Hence,

$$M\underline{c} = -\left[\int_B \underline{\rho}\, dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} (\underline{r}_i + \underline{w})\, ds_i \right] \tag{2.9$'$}$$

where M is the total mass of the vehicle. From the definition of the point O, it follows that

$$\int_B \underline{\rho}\, dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \underline{r}_i\, ds_i = 0 \tag{2.10}$$

so that Eq. (2.9)$'$ reduces to

$$M\underline{c} = -\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \underline{w}_i\, ds_i \tag{2.9$''$}$$

Now, denote by $\underline{\omega}$ the inertia angular velocity vector of the vector basis $[\underline{b}]$. Then, the inertia space time derivative $\dot{\underline{a}}$ ($\underline{a}$ is any vector) is given by

$$\dot{\underline{a}} = \mathring{\underline{a}} + \underline{\omega} \times \underline{a} \tag{2.11}$$

where $\mathring{\underline{a}}$ implies the time differentiation of the vector $\underline{a}$ with respect to the vector basis $[\underline{b}]$. . Using Eqs. (2.9) and (2.11), Eq. (2.8) reduces to

$$2T = -\frac{1}{M}\left[\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \mathring{\underline{w}}_i\, ds_i \cdot \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \mathring{\underline{w}}_i\, ds_i \right.$$
$$\left. + 2\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \mathring{\underline{w}}_i\, ds_i \cdot \left(\underline{\omega} \times \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \underline{w}_i\, ds_i\right)\right.$$

$$+ (\underline{\omega} \times \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \underline{w}_i ds_i) \cdot (\underline{\omega} \times \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \underline{w}_i ds_i)]$$

$$+ \int_B (\underline{\omega} \times \underline{\rho}) \cdot (\underline{\omega} \times \underline{\rho}) dm$$

$$+ \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \Big\{ \underline{\dot{w}}_i \cdot \underline{\dot{w}}_i + 2 \underline{\dot{w}}_i \cdot [\underline{\omega} \times (\underline{w}_i + \underline{r}_i)]$$

$$+ [\underline{\omega} \times (\underline{w}_i + \underline{r}_i)] \cdot [\underline{\omega} \times (\underline{w}_i + \underline{r}_i)] \Big\} ds_i . \qquad (2.12)$$

Let the vectors $\underline{\omega}$, $\underline{w}_i$ and $\underline{r}_i$ be written in expanded form as

$$\underline{\omega} = [\underline{b}]^T \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad \underline{r}_i = [\underline{a}_i]^T \begin{pmatrix} \zeta_i \\ 0 \\ 0 \end{pmatrix} \quad \underline{w}_i = [\underline{a}_i]^T \begin{pmatrix} 0 \\ w_{i2} \\ w_{i3} \end{pmatrix} \qquad (2.13)$$

where w_{i2} and w_{i3} are deflections in, and perpendicular to a plane normal to the spin axis, respectively. We shall here consider the case where the appendages undergo only small bending deformations. Hence, the quantities w_{i2} and w_{i3} are considered to be small. Moreover, we shall suppose that the variation of $\underline{\omega}$ from a nominal value $\underline{\omega}_n$ is small and the nominal value $\underline{\omega}_n$ is given by

$$\underline{\omega}_n = [\underline{b}]^T \begin{pmatrix} 0 \\ 0 \\ \omega_o \end{pmatrix} \qquad (2.14)$$

Then, ω_1 and ω_2 are small, and the relative changes in the variable ω_3 from its nominal value ω_o is also small. Substituting Eqs. (2.13) into Eq. (2.12) and taking only the terms up to the second order, we find

$$2T = I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2$$

$$-\frac{1}{2}\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}(\ell_i^2-\zeta_i^2)\left[\left(\frac{\partial w_{i2}}{\partial\zeta_i}\right)^2+\left(\frac{\partial w_{i3}}{\partial\zeta_i}\right)^2\right]\omega_3^2\,d\zeta_i$$

$$+\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}[\dot{w}_{i2}^2+\dot{w}_{i3}^2+2(S\gamma_i\zeta_i\dot{w}_{i3}\omega_1-C\gamma_i\zeta_i\dot{w}_{i3}\omega_2$$

$$+\zeta_i\dot{w}_{i2}\omega_3)-2(C\gamma_i\zeta_i w_{i3}\omega_3\omega_1+S\gamma_i\zeta_i w_{i3}\omega_3\omega_2)+w_{i2}^2\omega_3^2]\,ds_i$$

$$-\frac{1}{M}\Big\{\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)^2+\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)^2$$

$$+\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}\dot{w}_{i3}\,d\zeta_i\Big)^2$$

$$+2\Big[\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i w_{i2}\,d\zeta_i\Big)\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)$$

$$-\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i w_{i2}\,d\zeta_i\Big)\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)\Big]\omega_3$$

$$+\Big[\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i w_{i2}\,d\zeta_i\Big)^2+\Big(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i w_{i2}\,d\zeta_i\Big)^2\Big]\omega_3^2\Big\} \tag{2.15}$$

where I_1, I_2, I_3, are the moments of inertia of the system in the undeflected state about X_1, X_2, X_3 axes, respectively :

$$\left.\begin{aligned}
I_1 &= I_{B1}+\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S^2\gamma_i\zeta_i^2\,d\zeta_i = I_{B1}+\sum_{i=1}^{N}\frac{1}{3}\mu_i\ell_i^3S^2\gamma_i\\
I_2 &= I_{B2}+\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C^2\gamma_i\zeta_i^2\,d\zeta_i = I_{B2}+\sum_{i=1}^{N}\frac{1}{3}\mu_i\ell_i^3C^2\gamma_i\\
I_3 &= I_{B3}+\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}\zeta_i^2\,d\zeta_i = I_{B3}+\sum_{i=1}^{N}\frac{1}{3}\mu_i\ell_i^3
\end{aligned}\right\} \tag{2.16}$$

where I_{B1}, I_{B2}, I_{B3} are the moments of inertia of the rigid body B about X_1, X_2, X_3 axes, respectively. In this analysis, external forces are ignored. Hence, it follows that the potential energy of the system consists entirely of the elastic strain energy of the appendages. The elastic strain energy of the appendages, denoted by U, is, in the same approximation, given by

$$2U = \sum_{i=1}^{N} B_i \int_o^{\ell_i} [(\frac{\partial^2 w_{i2}}{\partial \zeta_i^2})^2 + (\frac{\partial^2 w_{i3}}{\partial \zeta_i^2})^2] d\zeta_i \tag{2.17}$$

where B_i is the bending stiffness of an appendage i.

Next, let us derive the system angular momentum expression. The system angular momentum $\underline{L}$ is given by

$$\underline{L} = \int_B (\underline{c} + \underline{\rho}) \times (\dot{\underline{c}} + \dot{\underline{\rho}}) dm + \sum_{i=1}^{N} \mu_i \int_o^{\ell_i} (\underline{c} + \underline{r}_i + \underline{w}_i)$$
$$\times (\dot{\underline{c}} + \dot{\underline{r}}_i + \dot{\underline{w}}_i) ds_i \,. \tag{2.18}$$

From the mass center definition, Eq. (2.9), and formula (2.11), Eq. (2.18) reduces to

$$\underline{L} = \frac{1}{M} [(\sum_{i=1}^{N} \mu_i \int_o^{\ell_i} \mathring{\underline{w}}_i ds_i) \times (\sum_{i=1}^{N} \mu_i \int_o^{\ell_i} \underline{w}_i ds_i)$$
$$+ (\underline{\omega} \times \sum_{i=1}^{N} \mu_i \int_o^{\ell_i} \underline{w}_i ds_i) \times (\sum_{i=1}^{N} \mu_i \int_o^{\ell_i} \underline{w}_i ds_i)]$$
$$+ \int_B \underline{\rho} \times (\underline{\omega} \times \underline{\rho}) dm + \sum_{i=1}^{N} \mu_i \int_o^{\ell_i} \Big\{ (\underline{w}_i + \underline{r}_i) \times \mathring{\underline{w}}_i$$
$$+ \underline{w}_i \times [\underline{\omega} \times (\underline{w}_i + \underline{r}_i)] + \underline{r}_i \times [\underline{\omega} \times (\underline{w}_i + \underline{r}_i)] \Big\} ds_i \tag{2.19}$$

Using the expressions (2.13) and Eq. (2.7), we can express $\underline{L}$ in the expanded form as

$$L_1 = I_1 \omega_1 + \sum_{i=1}^{N} \mu_i \int_o^{\ell_i} S\gamma_i \zeta_i \mathring{w}_{i3} d\zeta_i - \sum_{i=1}^{N} \mu_i \int_o^{\ell_i} C\gamma_i \zeta_i w_{i3} \omega_3 d\zeta_i \tag{2.20.a}$$

$$L_2 = I_2\omega_2 - \sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i - \sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\zeta_i w_{i3}\omega_3\,d\zeta_i$$

(2.20.b)

$$L_3 = I_3\omega_3 + \sum_{i=1}^{N}\mu_i\int_0^{\ell_i}\zeta_i\dot{w}_{i2}\,d\zeta_i - \sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\zeta_i w_{i3}\,d\zeta_i\,\omega_1$$

$$-\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\zeta_i w_{i3}\,d\zeta_i\,\omega_2 + \sum_{i=1}^{N}\mu_i\int_0^{\ell_i} w_{i2}^{\,2}\,\omega_3\,d\zeta_i$$

$$-\frac{1}{M}\Big\{\Big[\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i w_{i2}\,d\zeta_i\Big)^2 + \Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i w_{i2}\,d\zeta_i\Big)^2\Big]\omega_3$$

$$+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i w_{i2}\,d\zeta_i\Big)$$

$$-\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i w_{i2}\,d\zeta_i\Big)\Big\}$$

(2.20.c)

$$-\sum_{i=1}^{N}\mu_i\frac{1}{2}\int_0^{\ell_i}(\ell_i^2-\zeta_i^2)\Big[\Big(\frac{\partial W_{i2}}{\partial\zeta_i}\Big)^2+\Big(\frac{\partial W_{i3}}{\partial\zeta_i}\Big)^2\Big]\omega_3\,d\zeta_i\,.$$

2.4 Stability Analysis

Using the definition (2.3) and Eqs. (2.15), (2.17), the total energy H of the system is written in the form

$$2H = I_1\omega_1^{\,2} + I_2\omega_2^{\,2} + I_3\omega_3^{\,2} + \sum_{i=1}^{N}\mu_i\int_0^{\ell_i}\Big\{\dot{w}_{i2}^{\,2} + \dot{w}_{i3}^{\,2}$$

$$+ 2\,(S\gamma_i\zeta_i\dot{w}_{i3}\omega_1 - C\gamma_i\zeta_i\dot{w}_{i3}\omega_2 + \zeta_i\dot{w}_{i2}\omega_3)$$

$$+\frac{1}{2}(\ell_i^{\,2}-\zeta_i^{\,2})\Big[\Big(\frac{\partial w_{i2}}{\partial\zeta_i}\Big)^2+\Big(\frac{\partial w_{i3}}{\partial\zeta_i}\Big)^2\Big]\omega_3^{\,2} - w_{i2}^{\,2}\,\omega_3^{\,2}$$

$$-2\,(C\gamma_i\zeta_i w_{i3}\omega_1\omega_3 + S\gamma_i\zeta_i w_{i3}\omega_2\omega_3)\Big\}\,d\zeta_i$$

$$-\frac{1}{M}\Big\{\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)^2 + \Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)^2$$

$$+\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}\dot{w}_{i3}\,d\zeta_i\right)^2$$

$$+2\left[\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i w_{i2}\,d\zeta_i\right)\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i \dot{w}_{i2}\,d\zeta_i\right)\right.$$

$$\left.-\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i w_{i2}\,d\zeta_i\right)\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i \dot{w}_{i2}\,d\zeta_i\right)\right]\omega_3$$

$$\left.+\left[\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i w_{i2}\,d\zeta_i\right)^2+\left(\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i w_{i2}\,d\zeta_i\right)^2\right]\omega_3^{\,2}\right\}$$

$$+\sum_{i=1}^{N}B_i\int_o^{\ell_i}\left[\left(\frac{\partial^2 w_{i2}}{\partial\zeta_i^{\,2}}\right)^2+\left(\frac{\partial^2 w_{i3}}{\partial\zeta_i^{\,2}}\right)^2\right]d\zeta_i\,. \tag{2.21}$$

Since the system considered here is free of external forces, the angular momentum of the system is conserved. This reltaion can be used to eliminate the angular velocity component ω_3 from Eq. (2.21) : Solving for ω_3 from Eqs. (2.5) and (2.20) and putting it into Eq. (2.21), we obtain

$$2H=I_1\left(1-\frac{I_1}{I_3}\right)\omega_1^{\,2}+I_2\left(1-\frac{I_2}{I_3}\right)\omega_2^{\,2}$$

$$+2\left[\left(1-\frac{I_1}{I_3}\right)\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i\right.$$

$$\left.+\frac{I_1}{I_3}\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i\zeta_i w_{i3}\,d\zeta_i\,\omega_o\right]\omega_1$$

$$+2\left[-\left(1-\frac{I_2}{I_3}\right)\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}C\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i\right.$$

$$\left.+\frac{I_2}{I_3}\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}S\gamma_i\zeta_i w_{i3}\,d\zeta_i\,\omega_o\right]\omega_2$$

$$+\sum_{i=1}^{N}\mu_i\int_o^{\ell_i}\left(\dot{w}_{i2}^{\,2}+\dot{w}_{i3}^{\,2}\right)d\zeta_i$$

$$
-\frac{1}{I_3}\Big[\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i\Big)^2+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i\Big)^2
$$

$$
+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i}\zeta_i\dot{w}_{i2}\,d\zeta_i\Big)^2\Big]
$$

$$
-\frac{1}{M}\Big[\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)^2+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\dot{w}_{i2}\,d\zeta_i\Big)^2
$$

$$
+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i}\dot{w}_{i3}\,d\zeta_i\Big)^2\Big]
$$

$$
+\frac{2}{I_3}\Big[\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i\Big)\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\zeta_i w_{i3}\,d\zeta_i\Big)
$$

$$
-\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\zeta_i\dot{w}_{i3}\,d\zeta_i\Big)\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\zeta_i w_{i3}\,d\zeta_i\Big)\Big]\omega_o
$$

$$
+\sum_{i=1}^{N}B_i\int_0^{\ell_i}\Big[\Big(\frac{\partial^2 w_{i2}}{\partial\zeta_i^2}\Big)^2+\Big(\frac{\partial^2 w_{i3}}{\partial\zeta_i^2}\Big)^2\Big]d\zeta_i
$$

$$
+\sum_{i=1}^{N}\mu_i\int_0^{\ell_i}\Big\{\frac{1}{2}(\ell_i^2-\zeta_i^2)\Big[\Big(\frac{\partial w_{i2}}{\partial\zeta_i}\Big)^2+\Big(\frac{\partial w_{i3}}{\partial\zeta_i}\Big)^2\Big]\omega_o^2
$$

$$
-w_{i2}^2\omega_o^2\Big\}d\zeta_i
$$

$$
-\frac{1}{I_3}\Big[\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i\zeta_i w_{i3}\,d\zeta_i\Big)^2+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i\zeta_i w_{i3}\,d\zeta_i\Big)^2\Big]\omega_o^2
$$

$$
+\frac{1}{M}\Big[\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} S\gamma_i w_{i2}\,d\zeta_i\Big)^2+\Big(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i} C\gamma_i w_{i2}\,d\zeta_i\Big)^2\Big]\omega_o^2. \tag{2.22}
$$

In this derivation, we have used the relation that the variation of ω_3 from a nominal value ω_o is small.

The system considered here is a mechanical system with a pervasive damping. It can then be concluded, by virture of the stability theorem, that the condition that the total Energy H is positive definite is the necessary and sufficient condition for the asymptotic stability of the motion of this system. This means that if H is positive definite, the nominal motion is asymptotically stable and if H is sign variable or negative definite, the motion is unstable.

For H to be positive definite it is necessary and sufficient that

$$\left.\begin{array}{ll} 1) & I_3 - I_1 > 0 \\ 2) & I_3 - I_2 > 0 \\ 3) & \text{The functional } H_1,\ H_2,\ H_3 \text{ and } H_4 \text{ must be} \\ & \text{positive definite with the conditions,} \\ & w_{i2} = 0,\ w_{i3} = 0,\ \dfrac{\partial w_{i2}}{\partial \zeta_i} = 0,\ \dfrac{\partial w_{i3}}{\partial \zeta_i} = 0 \ \text{ for } \zeta_i = 0 \end{array}\right\} \tag{2.23}$$

where

$$2H_1 = \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \dot{w}_{i2}^{\,2} d\zeta_i - \frac{1}{I_3} \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \zeta_i \dot{w}_{i2} \right)^2$$
$$- \frac{1}{M} \left[\left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} S\gamma_i \dot{w}_{i2} d\zeta_i \right)^2 + \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} C\gamma_i \dot{w}_{i2} d\zeta_i \right)^2 \right] \tag{2.24.a}$$

$$2H_2 = \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \dot{w}_{i3}^{\,2} d\zeta_i - \frac{1}{I_1} \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} S\gamma_i \zeta_i \dot{w}_{i3} d\zeta_i \right)^2$$
$$- \frac{1}{I_2} \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} C\gamma_i \zeta_i \dot{w}_{i3} d\zeta_i \right)^2$$
$$- \frac{1}{M} \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \dot{w}_{i3} d\zeta_i \right)^2 \tag{2.24.b}$$

$$2H_3 = \sum_{i=1}^{N} B_i \int_0^{\ell_i} \left(\frac{\partial^2 w_{i2}}{\partial \zeta_i^2}\right)^2 d\zeta_i + \frac{1}{2} \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} (\ell_i^2 - \zeta_i^2) \left(\frac{\partial w_{i2}}{\partial \zeta_i}\right)^2 d\zeta_i \omega_o^2$$

$$- \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} w_{i2}^2 \, d\zeta_i \omega_o^2$$

$$+ \frac{1}{M} \left[\left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} S\gamma_i w_{i2} \, d\zeta_i \right)^2 + \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} C\gamma_i w_{i2} \, d\zeta_i \right)^2 \right] \omega_o^2 \tag{2.24.c}$$

$$2H_4 = \sum_{i=1}^{N} B_i \int_0^{\ell_i} \left(\frac{\partial^2 w_{i3}}{\partial \zeta_i^2}\right)^2 d\zeta_i + \frac{1}{2} \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} (\ell_i^2 - \zeta_i^2) \left(\frac{\partial w_{i3}}{\partial \zeta_i}\right)^2 d\zeta_i \omega_o^2$$

$$- \frac{1}{I_3 - I_1} \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} C\gamma_i \zeta_i w_{i3} \, d\zeta_i \right)^2 \omega_o^2$$

$$- \frac{1}{I_3 - I_2} \left(\sum_{i=1}^{N} \mu_i \int_0^{\ell_i} S\gamma_i \zeta_i w_{i3} \, d\zeta_i \right)^2 \omega_o^2 \, . \tag{2.24.d}$$

First, let us determine the sign characters of the functionals H_1 and H_2. These functionals can, by means of the following device, be reduced to suitable forms to determine the sign characters of the functionals. Let the elastic deformation vector $\underline{w}_i$ be extended in the rigid body B :

$$\underline{w}_B = [\underline{b}]^T \begin{bmatrix} -w_{B2} S\gamma_B \\ w_{B2} C\gamma_B \\ w_{B3} \end{bmatrix} \tag{2.25}$$

Moreover, we extend the vector $\underline{r}_i$ in the rigid body B as follows :

$$\underline{r}_B = [\underline{b}]^T \begin{bmatrix} \zeta_B C\gamma_B \\ \zeta_B S\gamma_B \\ 0 \end{bmatrix} \tag{2.26}$$

where

$$\zeta_B = (\rho_1^2 + \rho_2^2)^{\frac{1}{2}}$$

$$\tan\gamma_B = (\rho_2/\rho_1).$$

and

$$\underline{\rho} = [\underline{b}]^T \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$$

Then, the functionals H_1 and H_2 become

$$2H_1 = (\dot{w}_2\dot{w}_2) - \frac{1}{I_3}(\zeta\dot{w}_2)(\zeta\dot{w}_2) - \frac{1}{M}[(S\gamma\dot{w}_2)(S\gamma\dot{w}_2)$$

$$+ (C\gamma\dot{w}_2)(C\gamma\dot{w}_2)] \tag{2.27}$$

$$2H_2 = (\dot{w}_3\dot{w}_3) - \frac{1}{I_1}(S\gamma\zeta\dot{w}_3)(S\gamma\zeta\dot{w}_3) - \frac{1}{I_2}(C\gamma\zeta\dot{w}_3)(C\gamma\zeta\dot{w}_3)$$

$$- \frac{1}{M}(\dot{w}_3)(\dot{w}_3) \tag{2.28}$$

where

$$(f\ g) = \int_B f_B g_B \, dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} f_i g_i d\zeta_i \, .$$

On the other hand, using this notation, we can also write the total mass M in the form

$$M = \int_B dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} d\zeta_i = (1). \tag{2.29}$$

From Eqs. (2.16), the moments of inertia I_1, I_2, I_3 are written as

$$I_1 = (S\gamma\zeta S\gamma\zeta) + \int_B \rho_3^2 \, dm \tag{2.30.a}$$

$$I_2 = (C\gamma\zeta C\gamma\zeta) + \int_B \rho_3^2 \, dm \tag{2.30.b}$$

$$I_3 = (\zeta \ \zeta) \tag{2.30.c}$$

Furthermore, from the definition of the mass center (2.10), we have

$$\left.\begin{aligned} &(C\gamma\zeta) = 0 \\ &(S\gamma\zeta) = 0 \ . \end{aligned}\right\} \tag{2.31}$$

Since the body axes X_1, X_2, X_3 are fixed along the principal axes of the system in the underformed state,

$$\int_B S\gamma_B \zeta_B C\gamma_B \zeta_B \, dm + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} S\gamma_i \zeta_i C\gamma_i \zeta_i d\zeta_i = (S\gamma\zeta C\gamma\zeta) = 0 \ . \tag{2.32}$$

From the relations (2.31), it can be concluded that in order to show that H_1 is positive definite it is sufficient to show that the functionals H_{11}, H_{12} are positive definite where

$$H_{11} = (\dot{w}_2 \dot{w}_2) - \frac{1}{I_3}(\zeta \dot{w}_2)(\zeta \dot{w}_2)$$

$$H_{12} = (\dot{w}_2 \dot{w}_2) - \frac{1}{M}[\,(S\gamma \dot{w}_2)(S\gamma \dot{w}_2) + (C\gamma \dot{w}_2)(C\gamma \dot{w}_2)\,] \ .$$

The functional H_{11} is positive definite if the following inequality is satisfied (Appendix (1))

$$1 - \frac{1}{I_3}(\zeta \ \zeta) \geqq 0 \ . \tag{2.33}$$

For the positive definiteness of H_{12} it is required that (Appendix (2))

$$M - [\,(S\gamma S\gamma) - \frac{(S\gamma C\gamma)^2}{(C\gamma C\gamma)}\,] > 0$$

$$
\left.\begin{aligned}
& M - [\,(C\gamma C\gamma) - \frac{(S\gamma C\gamma)^2}{(S\gamma S\gamma)}\,] > 0 \\
& [\,M - (S\gamma S\gamma)\,]\,[\,M - (C\gamma C\gamma)\,] \geq (S\gamma C\gamma)^2
\end{aligned}\right\} \qquad (2.34)
$$

From Eqs. (2.29), (2.30.c), we find that these conditions are always satisfied. Then, the functional H_1 turns out to be positive definite.

Analogous statements can be made with regard to the functional H_2. It follows from relations (2.31), (2.32) that for H_2 to be positive definite it is sufficient to show that the following functionals are positive definite,

$$
\left.\begin{aligned}
& H_{21} = (\,\dot{w}_3 \dot{w}_3\,) - \frac{1}{M}(\dot{w}_3)(\dot{w}_3) \\
& H_{22} = (\,\dot{w}_3 \dot{w}_3\,) - \frac{1}{I_1}(S\gamma\zeta\dot{w}_3)(S\gamma\zeta\dot{w}_3) \\
& H_{23} = (\dot{w}_3 \dot{w}_3) - \frac{1}{I_2}(C\gamma\zeta\dot{w}_3)(C\gamma\zeta\dot{w}_3)\ .
\end{aligned}\right\} \qquad (2.35)
$$

The functionals H_{21}, H_{22}, H_{23} are positive definite if the following conditions are satisfied :

$$
\left.\begin{aligned}
& 1 - \frac{1}{M}(1) \geq 0 \\
& 1 - \frac{1}{I_1}(S\gamma\zeta S\gamma\zeta) \geq 0 \\
& 1 - \frac{1}{I_2}(C\gamma\zeta C\gamma\zeta) \geq 0\,.
\end{aligned}\right\} \qquad (2.36)
$$

From Eqs. (2.29), (2.30) we find that these conditions are also satisfied. Then, it can be concluded that the functional H_2 is positive definite.

It can easily be shown that the functional H_3 is positive definite : From a property of the Legendre functions which holds that

$$\int_o^{\ell_i} (\ell_i^2 - \zeta_i^2)\left(\frac{\partial w_{i2}}{\partial \zeta_i}\right)^2 d\zeta_i \geqq 2 \int_o^{\ell_i} w_{i2}^2 \, d\zeta_i \tag{2.37}$$

Eq. (2.24.c) then becomes

$$2H_3 \geqq \sum_{i=1}^{N} B_i \int_o^{\ell_i} \left(\frac{\partial^2 w_{i2}}{\partial \zeta_i^2}\right)^2 d\zeta_i + \frac{1}{M}\Big[\Big(\sum_{i=1}^{N} \mu_i \int_o^{\ell_i} S\gamma_i w_{i2}\, d\zeta_i\Big)^2$$

$$+ \Big(\sum_{i=1}^{N} \mu_i \int_o^{\ell_i} C\gamma_i w_{i2}\, d\zeta_i\Big)^2\Big]\omega_o^2 \geqq 0 . \tag{2.38}$$

We may now conclude that the conditions

1) $I_3 - I_1 > 0$ (2.39.a)

2) $I_3 - I_2 > 0$ (2.39.b)

3) The functional H_4 is positive definite (2.39.c)
with the conditions , $w_{i3} = 0$, $\dfrac{\partial w_{i3}}{\partial \zeta_i} = 0$ for $\zeta_i = 0$

are necessary and sufficient for the total energy H of the system to be positive definite ; we can conclude that conditions (2.39) are complete necessary and sufficient conditions for the stability for the attitude motion for this spacecraft model. The condition (2.39.c) has a simple physical meaning : Elastic vibrations of the appendages, induced by gyroscopic action, result in a change of the total energy storage and, on the other hand, decreases through the change of the moments of inertia. The condition (2.39.c) shows that, for stability, the change of the totalenergy due to elastic vibrations of appendages must be positive.

Finally, let us derive a sufficient condition for stability in a simple form. We shall now define a new testing functional $\hat{H}_4$ given by

$$2\hat{H}_4 = \sum_{i=1}^{N} B_i \int_o^{\ell_i} \left(\frac{\partial^2 w_{i3}}{\partial \zeta_i^2}\right)^2 d\zeta_i + \frac{1}{2}\sum_{i=1}^{N} \mu_i \int_o^{\ell_i} (\ell_i^2 - \zeta_i^2)\left(\frac{\partial w_{i3}}{\partial \zeta_i}\right)^2 d\zeta_i \omega_o^2$$

$$-\frac{1}{3}\left[\sum_{i=1}^{N}\frac{\ell_i^3}{I_3-I_1}\mu_i^2C^2\gamma_i+\frac{\ell_i^3}{I_3-I_2}\mu_i^2S^2\gamma_i\right]\int_0^{\ell_i}w_{i3}^2\,d\zeta_i\,\omega_o^2\,. \tag{2.40}$$

Recalling Schwartz's inequality, it is not difficult to show that

$$\left.\begin{aligned}&\left(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i}C\gamma_i\zeta_iw_{i3}\,d\zeta_i\right)^2\leqq\frac{\ell_i^3}{3}\sum_{i=1}^{N}\mu_i^2C^2\gamma_i\int_0^{\ell_i}w_{i3}^2\,d\zeta_i\\&\left(\sum_{i=1}^{N}\mu_i\int_0^{\ell_i}S\gamma_i\zeta_iw_{i3}\,d\zeta_i\right)^2\leqq\frac{\ell_i^3}{3}\sum_{i=1}^{N}\mu_i^2S^2\gamma_i\int_0^{\ell_i}w_{i3}^2\,d\zeta_i\end{aligned}\right\} \tag{2.41}$$

It follows, from this inequalities, that

$$\hat{H}_4\leqq H_4$$

Then, from the condition (2.39), the total energy H of the system is positive definite if the following conditions are satisfied,

1) $I_3-I_1>0$ (2.42.a)

2) $I_3-I_2>0$ (2.42.b)

3) The least vlaue of the functional $\hat{H}_4$ is positive with the conditions, $w_{i3}=0$, $\frac{\partial w_{i3}}{\partial\zeta_i}=0$ for $\zeta_i=0$. (2.42.c)

The least value of the functional $\hat{H}_4$ is given by the least value of the eigenvalues of the eigenvalue problems defined by

$$\left.\begin{aligned}&B_i\frac{d^4E_{in}(\zeta_i)}{d\zeta_i^4}-\frac{\mu_i\omega_o^2}{2}\frac{d}{d\zeta_i}\left[(\ell_i^2-\zeta_i^2)\left(\frac{dE_{in}(\zeta_i)}{d\zeta_i}\right)\right]\\&-\frac{\ell_i^3\omega_o^2}{3}\left(\frac{\mu_i^2C^2\gamma_i}{I_3-I_1}+\frac{\mu_i^2S^2\gamma_i}{I_3-I_2}\right)E_{in}(\zeta_i)-\mu_i\Omega_{in}^4E_{in}(\zeta_i)=0\\&\zeta_i=0;\quad E_{in}(\zeta_i)=0,\qquad\frac{dE_{in}(\zeta_i)}{d\zeta_i}=0\\&\zeta_i=\ell_i;\quad\frac{d^2E_{in}(\zeta_i)}{d\zeta_i^2}=0,\qquad\frac{d^3E_{in}(\zeta_i)}{d\zeta_i^3}=0\end{aligned}\right\} \tag{2.43}$$

where Ω_{in}^{4} are the eigenvalues associated with the normal modes $E_{in}(\zeta_i)$. On the other hand, the normal modes $F_{in}(\zeta_i)$ for the deflections of a rotating cantilever perpendicular to the spin axis are given by the following eigenvalue problems :[12]

$$
\left.
\begin{aligned}
& B_i \frac{d^4 F_{in}(\zeta_i)}{d\zeta_i^{\,4}} - \frac{\mu_i \omega_o^{\,2}}{2} \frac{d}{d\zeta_i} \left[(\ell_i^{\,2} - \zeta_i^{\,2}) \left(\frac{dF_{in}(\zeta_i)}{d\zeta_i} \right) \right] \\
& - \mu_i \Lambda_{in}^{4} F_{in}(\zeta_i) = 0 \\
& \zeta_i = 0 , \qquad F_{in}(\zeta_i) = 0 , \qquad \frac{dF_{in}(\zeta_i)}{d\zeta_i} = 0 \\
& \zeta_i = \ell_i ; \qquad \frac{d^2 F_{in}(\zeta_i)}{d\zeta_i^{\,2}} = 0 , \qquad \frac{d^3 F_{in}(\zeta_i)}{d\zeta_i^{\,3}} = 0
\end{aligned}
\right\} \qquad (2.44)
$$

where Λ_{in}^{4} are eigenvalues associated with the normal modes $F_{in}(\zeta_i)$. Combining Eqs. (2.43) and (2.44) and denoting by Λ_{i1}^{4} the lowest eigenvalue associated with the normal modes $F_{in}(\zeta_i)$, we get, from condition (2.42.c),

$$
3)' \quad \frac{\Lambda_{i1}^{4}}{\omega_o^{\,2}} > \frac{\ell_i^{3}}{3} \left[\frac{\mu_i C^2 \gamma_i}{(I_3 - I_1)} + \frac{\mu_i S^2 \gamma_i}{(I_3 - I_2)} \right] . \qquad (2.42.c)'
$$

Conditions (2.42) present sufficient conditions for H to be positive definite. Hence, the attitude motion is asymptotically stable if the conditions (2.42) are satisfied. It should be noted that, by contrast with the conditions (2.39), the evaluation of of criteria (2.42) requires much less numerical work : The criteria (2.42) require only the first natural frequencies of the rotating cantilevers.

To develop a better understanding of the problem, a symmetrical spacecraft with four equal appendages is investigated (Fig.2.2). The results for the criteria (2.39) are expressed in the form of a parameter chart, shown in Fig.2.3.

The parameter points in the shaded region denote unstable configurations and the parameter points in the unshaded region denote stable configurations. Figure 2.3 shows that as the bending stiffness of the appendages increases the size of the instability region decreases and, in the limit ($B \to \infty$), the stability criteria reduce to the maximum inertia axis criteria. Hence, it can be concluded that the influence of flexibility of the appendages is to destabilize some spacecraft configurations which are stable as rigid bodies. In the same figure, the results obtained from the conditions (2.42) are also shown for comparison. It should be noted that the sufficient conditions (2.42) yield satisfactory results.

2.5 Conclusions

Attitude stability is investigated for a freely spinning spacecraft with flexible appendages. The spacecraft is modeled as a rigid body having attached to it long flexible appendages lying in a plane normal to the spin axis. The Liapunov direct method is employed with the total energy of the system, constrained through the angular momentum integral, used as a Liapunov function. The approach to the stability problem is based on the functional analysis.

Necessary and sufficient conditions for the stability of the attitude motion are established for this spacecraft model. Furthermore, a sufficient condition is obtained in a simple form. This sufficient condition may have substantial utility in a preliminary design of this class of spacecraft because it gives a satisfactory result with little numerical work. The analysis reveals that the influence of elasticity of the appendages is to destabilize some spacecraft configurations which are stable as rigid bodies.

Appendix (1) Positive Definiteness of H_{11}

Using Schwarz's inequality, we have

$$H_{11} \geqq (\dot{w}_2 \dot{w}_2) - \frac{1}{I_3} (\zeta\,\zeta)(\dot{w}_2 \dot{w}_2) \tag{1}$$

where the sign of equality holds if w_2 is proportional to ζ. Since the elastic displacement w_2 is always zero in the body B and not proportional to ζ, the sign of equality must be ignored :

$$H_{11} > (\dot{w}_2 \dot{w}_2) - \frac{1}{I_3} (\zeta\,\zeta)(\dot{w}_2 \dot{w}_2) . \tag{1$'$}$$

From inerquality $(1)'$, it can be easily shown that H_{11} is positive definite if

$$1 - \frac{1}{I_3} (\zeta\,\zeta) \geqq 0 . \tag{2}$$

Appendix (2) Positive Definiteness of H_{12}

We define the functions

$$\Psi_1 = \frac{S\gamma}{(S\gamma S\gamma)^{\frac{1}{2}}}$$

$$\Psi_2 = \frac{C\gamma - \dfrac{(S\gamma C\gamma)}{(S\gamma S\gamma)} S\gamma}{\left[(C\gamma C\gamma) - \dfrac{(S\gamma C\gamma)^2}{(S\gamma S\gamma)}\right]^{\frac{1}{2}}} \tag{3}$$

The following relations between the functions Ψ_i are obtained

$$(\Psi_i \Psi_j) = \delta_{i,j} \tag{4}$$

where $\delta_{i,j}$ is Kronecker's delta. By virture of Bessel's inequality, we find that

$$(\dot{w}_2\, \dot{w}_2) \geqq \frac{(S\gamma\dot{w}_2)^2}{(S\gamma S\gamma)} + \frac{\left[\dfrac{(S\gamma C\gamma)}{(S\gamma S\gamma)}(S\gamma\dot{w}_2) - (C\gamma\dot{w}_2)\right]^2}{(C\gamma C\gamma) - \dfrac{(S\gamma C\gamma)^2}{(S\gamma S\gamma)}} \tag{5}$$

Since the sign of equality can also be neglected, for H_{12} to be positive definite, it is sufficient to show that the following functional $\hat{H}_{12}$ is positive semidefinite :

$$\begin{aligned}\hat{H}_{12} = {} & \left[\frac{1}{(S\gamma S\gamma) - \dfrac{(S\gamma C\gamma)^2}{(C\gamma C\gamma)}} - \frac{1}{M}\right](S\gamma\dot{w}_2)^2 \\ & - \frac{2\dfrac{(S\gamma C\gamma)}{(C\gamma C\gamma)}}{(S\gamma S\gamma) - \dfrac{(S\gamma C\gamma)^2}{(C\gamma C\gamma)}}(S\gamma\dot{w}_2)(C\gamma\dot{w}_2) \\ & + \left[\frac{1}{(C\gamma C\gamma) - \dfrac{(S\gamma C\gamma)^2}{(S\gamma S\gamma)}} - \frac{1}{M}\right](C\gamma\dot{w}_2)^2 .\end{aligned} \tag{6}$$

If

$$\left.\begin{aligned} & M - \left[(S\gamma S\gamma) - \frac{(S\gamma C\gamma)^2}{(C\gamma C\gamma)}\right] > 0 \\ & M - \left[(C\gamma C\gamma) - \frac{(S\gamma C\gamma)^2}{(S\gamma S\gamma)}\right] > 0 \\ & [M - (S\gamma S\gamma)][M - (C\gamma C\gamma)] \geqq (S\gamma C\gamma)^2 \end{aligned}\right\} \tag{7}$$

$\hat{H}_{12}$ is positive semidefinite. Then, we find that H_{12} is positive definite if inequalities (7) are satisfied.

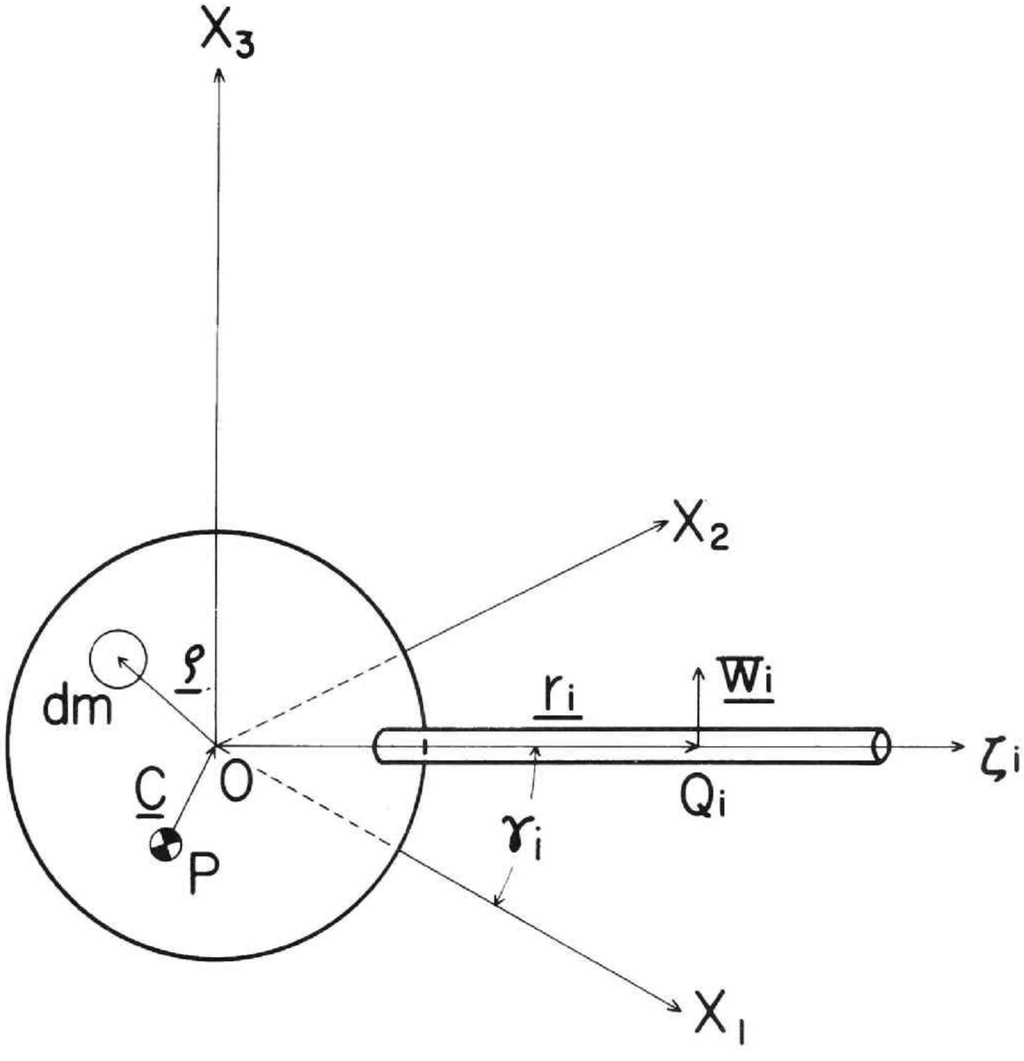

Fig. 2.1 Spacecraft configuration

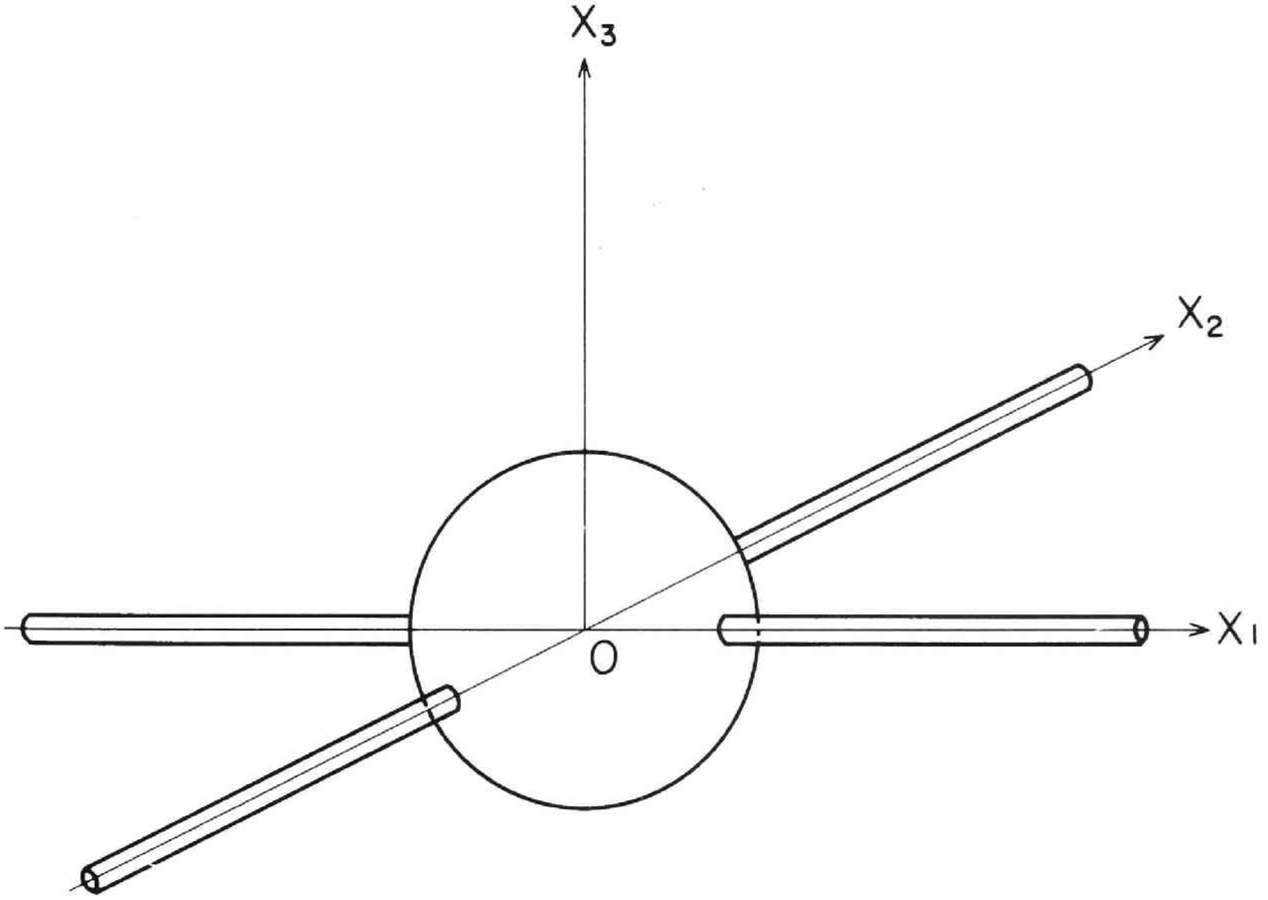

Fig. 2.2 Symmetrical spacecraft configuration

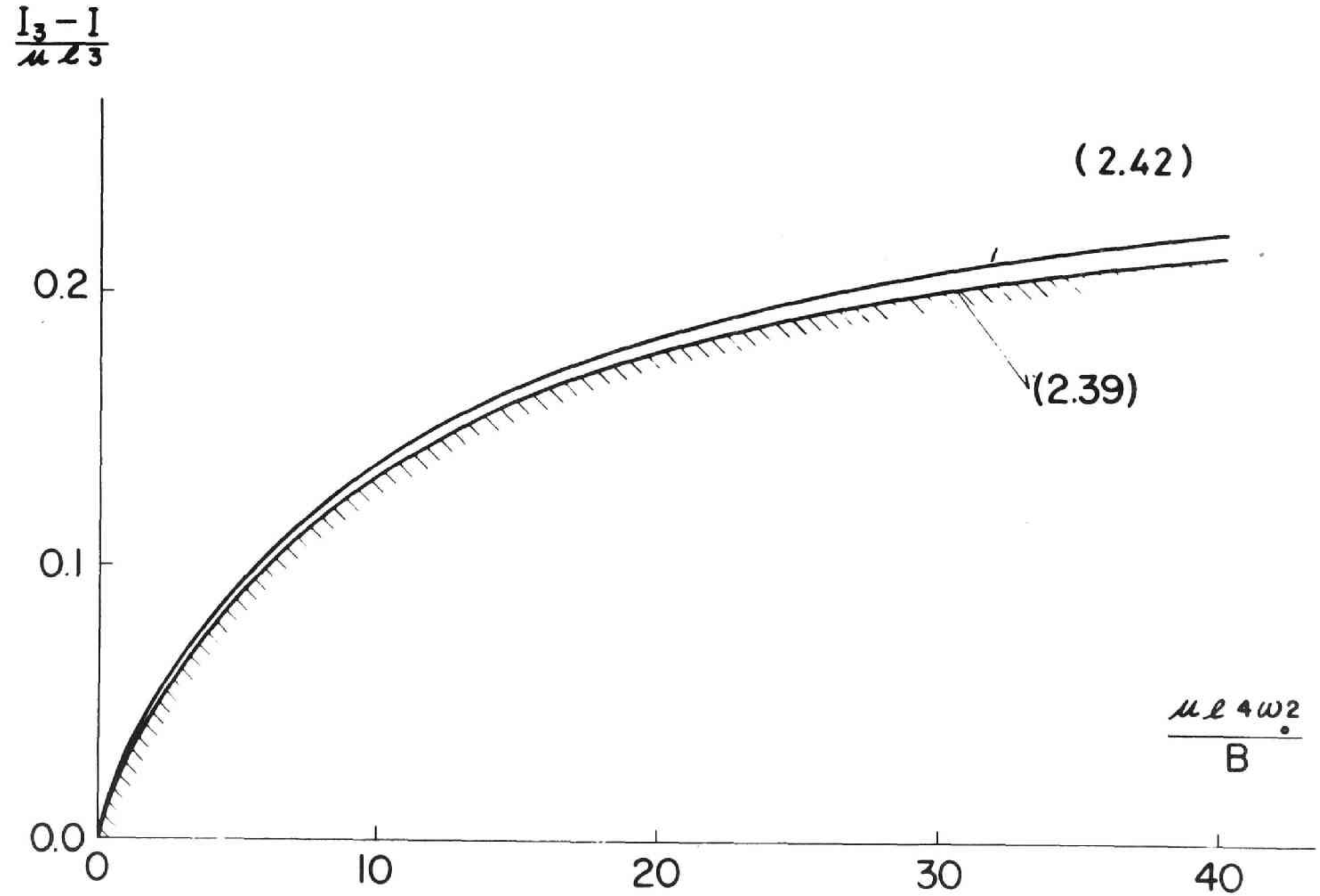

Fig. 2.3 Stability regions (hatched regions are unstable)

CHAPTER III

STABILITY AND PERFORMANCE OF A SPIN STABILIZED SPACECRAFT HAVING FLEXIBLE APPENDAGES

3.1 Introduction

The influence of flexible appendages on the attitude motion of a spinning spacecraft has been examined by many authors. In most cases, [2,3,4,5] the energy sink method is employed in the analysis. The basic assumption of this method is that the moments of inertia do not vary significantly and the angular momentum of the relative motion is negligible compared to the rigid body motion. The relative motion within a spacecraft is then idealized as an energy removal device and the rate of energy dissipation can be related to the change in nutational body motions. With this relationship an estimate of the motion of the spacecraft is easily obtained. Because of its analytical nature, this method gives a clear picture of the behavior of a freely spinning spacecraft with damping. However, this method is not appropriate in application to a spin stabilized spacecraft having large flexible appendages : Effects of flexibility of the appendages, in this case, become of prime importance on the attitude motion of the spacecraft, and the essential assumption of this method no longer holds good.

New approaches suitable for exploring the basic characteristics of the dynamics of this class of spacecraft must be developed from an analytical viewpoint. In this chapter, the method of averaging is employed to accomplish this goal. The spacecraft is modeled as a rigid body having attached to it flexible appendages lying in a plane which is normal to the spin axis.

First, linearized equations of motion are derived for this class of spacecraft rotating freely in space. The approach to the derivation of the equations of motion is based on the hybrid coordinate method [31] : Motions are described in terms of the angular velocity of the spacecraft and modal deformation coordinates for the appendages. Then, these equations are solved by using an analytical method which utilizes the method of averaging and closed form approximate solutions are obtained. The damping ratio and the frequency of the nutational body motion are derived from the solution. Attitude stability criteria are also obtained from the sign properties of the damping ratio. These results are compared with results obtained by the digital computer eigenvalue analysis.

3.2 Equations of Motion

Let us consider a spinning spacecraft composed of a heavy central rigid body and N light weight appendages, as shown in Fig. 3.1. We define the reference axes (X_1, X_2, X_3) which coincide with the principal axes of the system in the undeformed configuration. The X_3 axis is taken to be coincident with the spin axis. For an appendage i, we take an axis system (ζ_i, η_i, ξ_i) so that the appendage i coincides with the ζ_i axis when it is undeflected and the ξ_i axis coincides with the X_3 axis. The axes X_1 and ζ_i make an angle γ_i. Let the angular velocity of the (X_1, X_2, X_3) reference frame be defined by (ω_1, ω_2, ω_3) in the (X_1, X_2, X_3) reference frame. Let elastic deflections of an arbitrary point on an appendage i, be denoted by u_i and v_i where u_i and v_i are respectively, deflections in and perpendicular to a plane which is normal to the spin axis.

In the present study, we assume that the mass center of the toal configuration remains fixed at the mass center of the undeformed configuration. Then, the total kinetic energy T can be written in the form

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \Big\{ (\dot{u}_i^2 + \dot{v}_i^2) - 2\omega_1$$

$$[\, C\gamma_i(\dot{u}_i v_i - \dot{v}_i u_i) - S\gamma_i \zeta_i \dot{v}_i \,] - 2\omega_2 [\, S\gamma_i(\dot{u}_i v_i -$$

$$\dot{v}_i u_i) + C\gamma_i \zeta_i \dot{v}_i \,] + 2\omega_3 \zeta_i \dot{u}_i + \omega_1^2 (v_i^2 + C^2\gamma_i u_i^2$$

$$+ S2\gamma_i \zeta_i u_i) + \omega_2^2 (v_i^2 + S^2\gamma_i u_i^2 - S2\gamma_i \zeta_i u_i) + \omega_3^2 u_i^2$$

$$+ 2\omega_1\omega_2 (\frac{1}{2} S2\gamma_i u_i^2 - C2\gamma_i \zeta_i u_i) - 2\omega_2\omega_3 (C\gamma_i u_i v_i$$

$$+ S\gamma_i \zeta_i v_i) + 2\omega_1\omega_3 (S\gamma_i u_i v_i - C\gamma_i \zeta_i v_i) \Big\} ds_i \qquad (3.1)$$

where I_1, I_2, I_3 are the moments of inertia of the system about the X_1, X_2, X_3 axes, respectively, μ_i the mass per unit length of an appendage i, ds_i the arc length along the appendage i, ℓ_i the total length of the appendage i, and $C\gamma_i = \cos\gamma_i$,...... .., $S2\gamma_i = \sin\ 2\gamma_i$.

We shall, for the moment, confine ourselves to the problem of small changes in the attitude of the spacecraft. Since departures from a steady spinning state are small, the angular velocity components ω_1, ω_2 are small. For the same reason, deflections u_i, v_i are also small. Then the kinetic energy expression, on neglecting terms of higher order than the second, becomes

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \Big\{ (\dot{u}_i^2 + \dot{v}_i^2) + 2\omega_1 S\gamma_i \zeta_i \dot{v}_i$$

$$- 2\omega_2 C\gamma_i \zeta_i \dot{v}_i + 2\omega_3 \zeta_i \dot{u}_i + \omega_3^2 [\, u_i^2 - \frac{1}{2}(\ell_i^2 - \zeta_i^2)\, ((\frac{\partial u_i}{\partial \zeta_i})^2$$

$$+ (\frac{\partial v_i}{\partial \zeta_i})^2)\,] - 2\omega_2\omega_3 S\gamma_i \zeta_i v_i - 2\omega_1\omega_3 C\gamma_i \zeta_i v_i \Big\} d\zeta_i \,. \qquad (3.2)$$

Attitude motions of a spinning spacecraft are affected by external forces of various forms. Generally, these forces are so small that it can be justified that the system is considered as moment-free for a short time period. Ignoring external forces, we find that the potential energy U of the system consists entirely of the elastic strain energy of the appendages, i.e.,

$$2U = \sum_{i=1}^{N} B_i \int_0^{\ell_i} \left[\left(\frac{\partial^2 u_i}{\partial \zeta_i^2} \right)^2 + \left(\frac{\partial^2 v_i}{\partial \zeta_i^2} \right)^2 \right] d\zeta_i \tag{3.3}$$

where B_i is the bending stiffness of an appendage i.

The energy dissipation which results from elastic deformations of the appendages is represented by Rayleigh's dissipation function F, which is given by

$$F = \sum_{i=1}^{N} \mu_i \delta_i \int_0^{\ell_i} \left[\left(\dot{u}_i^2 + \dot{v}_i^2 \right) \right] d\zeta_i \tag{3.4}$$

where δ_i is the damping ratio of an appendage i.

In order to perform a dynamics analysis on the basis of the modal analysis, we represent elastic deformations by the following series :

$$\left. \begin{aligned} u_i &= \ell_i \sum_{n=1}^{\infty} P_{in}(t) E_n(\hat{\zeta}) \\ v_i &= \ell_i \sum_{n=1}^{\infty} T_{in}(t) E_n(\hat{\zeta}) \end{aligned} \right\} \tag{3.5}$$

where $E_n\ (\hat{\zeta})$ are normal modes associated with a cantilever and $P_{in}(t)$, $T_{in}(t)$ are the corresponding generalized coordinates. The normal modes $E_n(\hat{\zeta})$ satisfy the differential equations

$$\frac{d^4 E_n(\hat{\zeta})}{d\hat{\zeta}^4} - \lambda_n^4 E_n(\hat{\zeta}) = 0 \tag{3.6}$$

and boundary conditions

$$\left.\begin{aligned} &\hat{\zeta} = 0, \quad E_n(\hat{\zeta}) = 0, \quad \frac{dE_n(\hat{\zeta})}{d\hat{\zeta}} = 0 \\ &\hat{\zeta} = 1, \quad \frac{d^2 E_n(\hat{\zeta})}{d\hat{\zeta}^2} = 0, \quad \frac{d^3 E_n(\hat{\zeta})}{d\hat{\zeta}^3} = 0 \end{aligned}\right\} \tag{3.7}$$

where λ_n^4 are the eigenvalues of the normal modes $E_n(\hat{\zeta})$ and $\hat{\zeta} = \frac{\zeta_i}{\ell_i}$. In addition, they are normalized such that

$$\int_o^1 E_n(\hat{\zeta})\, E_m(\hat{\zeta})\, d\hat{\zeta} = \delta_{n,m}$$

where $\delta_{n,m}$ is Kronecker's delta. As shown later, the appendages are excited below the first natural frequencies, so that we can truncate the series expansion at the first mode, i.e., n=1. Substituting Eqs. (3.5) into Eqs. (3.2), (3.3), (3.4), truncating at n=1 and neglecting the suffix 1, we obtain

$$\begin{aligned} 2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \sum_{i=1}^{N} \mu_i \ell_i^3 \Big\{ &(\dot{P}_i^2 + \dot{T}_i^2) + 2\omega_1 S\gamma_i \epsilon \dot{T}_i \\ &- 2\omega_2 C\gamma_i \epsilon \dot{T}_i + 2\omega_3 \epsilon \dot{P}_i + \omega_3^2 [\, P_i^2 - \beta(P_i^2 + T_i^2)\,] \\ &- 2\omega_2\omega_3 S\gamma_i \epsilon T_i - 2\omega_1\omega_3 \epsilon T_i \Big\} \end{aligned} \tag{3.8}$$

$$2U = \sum_{i=1}^{N} \left(\frac{\lambda^4 B_i}{\ell_i}\right) (P_i^2 + T_i^2) \tag{3.9}$$

$$F = \sum_{i=1}^{N} \mu_i \ell_i^3 \delta_i (\dot{P}_i^2 + \dot{T}_i^2) \tag{3.10}$$

where

$$\epsilon = \int_o^1 \hat{\zeta} E(\hat{\zeta})\, d\hat{\zeta} = -0.5688$$

$$\beta = \frac{1}{2}\int_0^1 (1-\hat{\zeta}^2)\left(\frac{dE(\hat{\zeta})}{d\hat{\zeta}}\right)^2 d\hat{\zeta} = 1.193$$

The Lagrange equations of motion for the generalized coordinates P_i, T_i are written in the forms [32]

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{d\dot{P}_i}\right) - \left(\frac{\partial L}{\partial P_i}\right) &= -\left(\frac{\partial F}{\partial \dot{P}_i}\right) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{T}_i}\right) - \left(\frac{\partial L}{\partial T_i}\right) &= -\left(\frac{\partial F}{\partial \dot{T}_i}\right) \end{aligned}\right\} \tag{3.11}$$

where Lagrangian L is given by

$$L = T - U . \tag{3.12}$$

Since the coordinates ω_1, ω_2, ω_3 are so called quasi-coordinates, the corresponding equations of motion are given by [32]

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + \omega_2\left(\frac{\partial T}{\partial \omega_3}\right) - \omega_3\left(\frac{\partial T}{\partial \omega_2}\right) &= N_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) + \omega_3\left(\frac{\partial T}{\partial \omega_1}\right) - \omega_1\left(\frac{\partial T}{\partial \omega_3}\right) &= N_2 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) + \omega_1\left(\frac{\partial T}{\partial \omega_2}\right) - \omega_2\left(\frac{\partial T}{\partial \omega_1}\right) &= N_3 \end{aligned}\right\} \tag{3.13}$$

where N_1, N_2, N_3 are the torque components about the X_1, X_2, X_3 axes, respectively. Since external torques are neglected, it follows that

$$N_1 = N_2 = N_3 = 0 . \tag{3.14}$$

Substitution of Eqs. (3.8), (3.9), (3.10), (3.14) into Eqs. (3.11), (3.13) leads to the linearized equations of motion as follows :

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_o\omega_2 = -\sum_{i=1}^{N} \mu_i \ell_i^3 \epsilon S \gamma_i (\ddot{T}_i + \omega_o^2 T_i) \tag{3.15.a}$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_o\omega_1 = \sum_{i=1}^{N} \mu_i \ell_i^3 \epsilon C\gamma_i(\ddot{T}_i + \omega_o^2 T_i) \tag{3.15.b}$$

$$\ddot{T}_i + 2\delta_i\dot{T}_i + k_{Ti}^2 T_i = C\gamma_i\epsilon(\dot{\omega}_2 - \omega_o\omega_1) - S\gamma_i\epsilon(\dot{\omega}_1 + \omega_o\omega_2) \qquad (i=1,\ldots\ldots, N) \tag{3.15.c}$$

$$I_3\dot{S} + \sum_{i=1}^{N} \mu_i \ell_i^3 \epsilon \ddot{P}_i = 0 \tag{3.15.d}$$

$$\ddot{P}_i + 2\delta_i\dot{P}_i + k_{Pi}^2 P_i = -\epsilon\dot{S} \qquad (i=1,\ldots\ldots, N) \tag{3.15.e}$$

where

$$k_{Pi}^2 = \frac{\lambda^4 B_i}{\mu_i \ell_i^4} + (\beta - 1)\omega_o^2$$

$$k_{Ti}^2 = \frac{\lambda^4 B_i}{\mu_i \ell_i^4} + \beta\omega_o^2 .$$

In this derivation, we have assumed that

$$\omega_3 = \omega_o + S \tag{3.16}$$

where ω_o is a constant spin velocity and S is a small variation. The linearized equations of motion fall into two uncoupled sets : the first set (Eqs. (3.15.a), (3.15.b), (3.15.c)). describes nutational body motions and appendage deformations perpendicular to a spin plane, and the second set (Eqs. (3.15.d), (3.15.e)) describes the change in the spin velocity and appendage deformations in a spin plane. The second set is independent of nutational body motions, and so is not of direct interest in the present analysis. For this reason, in the remainder of this chapter the first set is further studied.

3.3 Analysis

Solving Eqs. (3.15.c), we obtain

$$T_i = \frac{\epsilon}{2i\bar{k}_{Ti}}\int^t \left[C\gamma_i(\dot{\omega}_2 - \omega_o\omega_1) - S\gamma_i(\dot{\omega}_1 + \omega_o\omega_2) \right]$$

$$\times e^{(i\hat{k}_{Ti}-\delta_i)(t-t')} dt'$$

$$+ \text{complex conjugate part}$$

$$\equiv \epsilon L_{ir}(\omega_1, \omega_2, t) \tag{3.17}$$

where

$$\hat{k}_{Ti} = \sqrt{k_{Ti}^2 - \delta_i^2} \; .$$

Substituting these expressions for T_i into Eqs. (3.15.a) (3.15.b), we obtain the following integro differential equations :

$$\left.\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_o\omega_2 &= -\epsilon^2 \sum_{i=1}^{N} S\gamma_i \mu_i \ell_i^3 \left(\frac{d^2}{dt^2} + \omega_o^2\right) L_{ir}(\omega_1, \omega_2, t) \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_o\omega_1 &= \epsilon^2 \sum_{i=1}^{N} C\gamma_i \mu_i \ell_i^3 \left(\frac{d^2}{dt^2} + \omega_o^2\right) L_{ir}(\omega_1, \omega_2, t) \; . \end{aligned}\right\} \tag{3.18}$$

Equations (3.18) suggest to us the applicability of the method of averaging [36] to this problem. The first step in the procedure is to transform Eqs. (3.18) into a form suitable for application of the method of averaging. We shall introduce a new variable a by the following transformation :

$$\left.\begin{aligned} \omega_1 &= \hat{\omega}_1 (a e^{i\hat{\alpha} t} + a^* e^{-i\hat{\alpha} t}) \\ \omega_2 &= -i\hat{\omega}_2 (a e^{i\hat{\alpha} t} - a^* e^{-i\hat{\alpha} t}) \end{aligned}\right\} \tag{3.19}$$

where

$$\hat{\alpha} = \omega_o \left[\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right) \right]^{\frac{1}{2}}$$

$$\hat{\omega}_1 = \frac{1}{I_1}\left(\frac{I_3}{I_2} - 1\right)^{\frac{1}{2}}, \quad \hat{\omega}_2 = \frac{1}{I_2}\left(\frac{I_3}{I_1} - 1\right)^{\frac{1}{2}},$$

a* is a complex conjugate of a. It may be noted that the following necessary condition for stability is deduced from the condition that $\hat{\alpha}$ is real :

$$\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right) > 0 . \tag{3.20}$$

The equations (3.17), on substitution of Eqs. (3.19), become

$$T_i = \frac{\epsilon}{2i\hat{k}_{Ti}} \int^t [\, f_{1i}\, a e^{i\hat{\alpha}t} + f_{1i}^{\,*} a^* e^{-i\hat{\alpha}t} + f_{2i}\, \dot{a} e^{i\hat{\alpha}t}$$

$$+ f_{2i}^{\,*} \dot{a}^* \,]\, e^{(i\hat{k}_{Ti} - \delta_i)(t-t')}\, dt' + \text{complex conjugate part}$$

$$= \epsilon L_{ir}(a, a^*, t) \tag{3.21}$$

where

$$f_{1i} = C\gamma_i(\hat{\omega}_2\hat{\alpha} - \omega_o\hat{\omega}_1) - iS\gamma_i(\hat{\omega}_1\hat{\alpha} - \omega_o\hat{\omega}_2)$$

$$f_{2i} = -\, S\gamma_i\hat{\omega}_1 - iC\gamma_i\hat{\omega}_2 \, .$$

Substituting this expression and Eqs. (3.19) into Eqs. (3.18), we find

$$\dot{a} = -\frac{1}{2}\sum_{i=1}^{N} \mu_i \ell_i^{\,3} \epsilon^2 \left(\frac{S\gamma_i}{I_1\hat{\omega}_1} - \frac{iC\gamma_i}{I_2\hat{\omega}_2}\right)\left(\frac{d^2}{dt^2} + \omega_o^{\,2}\right)$$

$$\times L_{ir}(a, a^*, t) e^{-i\hat{\alpha}t} \, . \tag{3.22}$$

The next step is to expand the variable a in the power series of ϵ^2. We assume that

$$a = \hat{a} + \sum_{n=1}^{\infty} \epsilon^{2n} F_{\hat{a}}^{(n)}(\hat{a}, \hat{a}^*) F_t^{(n)}(t) \tag{3.23.a}$$

$$\dot{\hat{a}} = \sum_{n=1}^{\infty} \epsilon^{2n} G_{\hat{a}}^{(n)} (\hat{a}, \hat{a}^*) \; . \qquad (3.23.b)$$

On substituting Eqs. (3.23) into Eq. (3.21) and carrying out the integration by parts, the function $L_{ir}(a, a^*, t)$ is expressed as the power series in ϵ^2:

$$L_{ir} = \frac{1}{2i\hat{k}_{Ti}} (h_i f_{1i} \hat{a} e^{i\hat{\alpha}t} - h_i^* f_{1i}^* \hat{a}^* e^{-i\hat{\alpha}t})$$

$$+ \frac{\epsilon^2}{2i\hat{k}_{Ti}} [(h_i f_{2i} G_{\hat{a}}^{(1)} e^{i\hat{\alpha}t} - h_i^* f_{2i}^* G_{\hat{a}}^{(1)*} e^{-i\hat{\alpha}t})$$

$$-(g_i f_{1i} G_{\hat{a}}^{(1)} e^{i\hat{\alpha}t} - g_i^* f_{1i}^* G_{\hat{a}}^{(1)*} e^{-i\hat{\alpha}t})]$$

$$- \frac{\epsilon^2}{\hat{k}_{Ti}} [f_{1i} F_{\hat{a}}^{(1)} \int^t F_t^{(1)} e^{\delta_i (t'-t)} e^{i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t)dt'$$

$$+ f_{1i}^* F_{\hat{a}}^{(1)*} \int^t F_t^{(1)*} e^{\delta_i (t'-t)} e^{-i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t)dt'$$

$$+ f_{2i} F_{\hat{a}}^{(1)} \int^t \dot{F}_t^{(1)} e^{\delta_i (t'-t)} e^{i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t)dt'$$

$$+ f_{2i}^* F_{\hat{a}}^{(1)*} \int^t \dot{F}_t^{(1)*} e^{\delta_i (t'-t)} e^{-i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t)dt']$$

$$(3.24)$$

where

$$h_i = h_{i1} + ih_{i2}$$

$$g_i = g_{i1} + ig_{i2}$$

$$h_{i1} = \frac{\delta_i}{\delta_i^2 + (\hat{k}_{Ti} - \hat{\alpha})^2} - \frac{\delta_i}{\delta_i^2 + (\hat{k}_{Ti} + \hat{\alpha})^2}$$

$$h_{i2} = \frac{(\hat{k}_{Ti} - \hat{\alpha})}{\delta_i^2 + (\hat{k}_{Ti} - \hat{\alpha})^2} + \frac{(\hat{k}_{Ti} + \hat{\alpha})}{\delta_i^2 + (\hat{k}_{Ti} + \hat{\alpha})^2}$$

$$g_{i1} = \frac{\delta_i^2 - (\hat{k}_{Ti} - \hat{\alpha})^2}{[\,\delta_i^2 + (\hat{k}_{Ti} - \hat{\alpha})^2\,]^2} - \frac{\delta_i^2 - (\hat{k}_{Ti} + \hat{\alpha})^2}{[\,\delta_i^2 + (\hat{k}_{Ti} + \hat{\alpha})^2\,]^2}$$

$$g_{i2} = \frac{2\delta_i(\hat{k}_{Ti} - \hat{\alpha})}{[\,\delta_i^2 + (\hat{k}_{Ti} - \hat{\alpha})^2\,]^2} + \frac{2\delta_i(\hat{k}_{Ti} + \hat{\alpha})}{[\,\delta_i^2 + (\hat{k}_{Ti} + \hat{\alpha})^2\,]^2} .$$

Substituting Eq. (3.24) into Eq. (3.22) and equating the terms of the same order of ϵ^2, we obtain a series of equations as follows :

$$G_{\hat{a}}^{(1)} + F_{\hat{a}}^{(1)} \dot{F}_t^{(1)}$$

$$= -\frac{1}{2} \sum_{i=1}^{N} \mu_i \ell_i^3 \left(\frac{S\gamma_i}{I_1 \hat{\omega}_1} - \frac{iC\gamma_i}{I_2 \hat{\omega}_2} \right) \frac{(\omega_o^2 - \hat{\alpha}^2)}{(2i\hat{k}_{Ti})} (f_{1i} h_i \hat{a}$$

$$- f_{1i}^* h_i^* \hat{a}^* e^{-2i\hat{\alpha} t}) \tag{3.25.a}$$

$$G_{\hat{a}}^{(2)} + F_{\hat{a}}^{(2)} \dot{F}_t^{(2)}$$

$$= -\left(G_{\hat{a}}^{(1)} \frac{\partial F_{\hat{a}}^{(1)}}{\partial \hat{a}} F_t^{(1)} + G_{\hat{a}}^{(1)*} \frac{\partial F_{\hat{a}}^{(1)}}{\partial \hat{a}^*} F_t^{(1)} \right)$$

$$-\frac{1}{2} \sum_{i=1}^{N} \mu_i \ell_i^3 \left(\frac{S\gamma_i}{I_1 \hat{\omega}_1} - \frac{iC\gamma_i}{I_2 \hat{\omega}_2} \right) \left[\frac{(\omega_o^2 - \hat{\alpha}^2)}{(2i\hat{k}_{Ti})} (f_{2i} h_i G_{\hat{a}}^{(1)} \right.$$

$$- f_{2i}^* h_i^* G_{\hat{a}}^{(1)*} e^{-2i\hat{\alpha} t}) - \frac{(\omega_o^2 - \hat{\alpha}^2)}{(2i\hat{k}_{Ti})} (g_i f_{1i} G_{\hat{a}}^{(1)} .$$

$$-g_i^* f_{1i}^* G_{\hat{a}}^{(1)*} e^{-2i\hat{\alpha}t}) + \frac{\hat{\alpha}}{\hat{k}_{Ti}} (f_{1i} h_i G_{\hat{a}}^{(1)} + f_{1i}^* h_i^* G_{\hat{a}}^{(1)*} e^{-2i\hat{\alpha}t})]$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \mu_i \ell_i^3 \left(\frac{S\gamma_i}{I_1 \hat{\omega}_1} - \frac{iC\gamma_i}{I_2 \hat{\omega}_2} \right) \frac{1}{\hat{k}_{Ti}} \left(\omega_o^2 + \frac{\partial^2}{\partial t^2} \right)$$

$$[f_{1i} F_{\hat{a}}^{(1)} \int^t F_t^{(1)} e^{\delta_i (t'-t)} e^{i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t) dt'$$

$$+ f_{1i}^* F_{\hat{a}}^{(1)*} \int^t F_t^{(1)*} e^{\delta_i (t'-t)} e^{-i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t)\, dt'$$

$$+ f_{2i} F_{\hat{a}}^{(1)} \int^t \dot{F}_t^{(1)} e^{\delta_i (t'-t)} e^{i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t) dt'$$

$$+ f_{2i}^* F_{\hat{a}}^{(1)*} \int^t \dot{F}_t^{(1)*} e^{\delta_i (t'-t)} e^{-i\hat{\alpha}t'} \sin\hat{k}_{Ti}(t'-t)\, dt'] . \tag{3.25.b}$$

. . .

It is noted that there are two unknowns, $G_{\hat{a}}^{(n)}$ and $\dot{F}_{\hat{a}}^{(n)} F_t^{(n)}$, in a single equation of the each order approximation. Hence, some additional conditions are needed. The crucial point of the method of averaging consists in the effective use of an averaging operation to this end. In the present analysis, the following averaging operation may suitably be chosen :

$$\overline{\Phi} (\hat{a}, \hat{a}^*) = \lim_{T \to \infty} \frac{1}{T} \int_o^T \Phi (\hat{a}, \hat{a}^*, t)\, dt \tag{3.26}$$

where $\Phi (a, \hat{a}^*, t)$ is any function of $\hat{a}$, $\hat{a}^*$ and t. With this choice, the n-th order equation is, then, divided into two as follows :

$$G_{\hat{a}}^{(n)} = \overline{RHS} \tag{3.27.a}$$

$$F_{\hat{a}}^{(n)} \dot{F}_t^{(n)} = RHS - \overline{RHS} \tag{3.27.b}$$

Eqs. (3.27.b) are the first order differential equations, some additional conditions are necessary. These conditions we take to be

$$\overline{F_{\hat{a}}^{(n)} F_t^{(n)}} = 0 \ . \tag{3.28}$$

Application of this procedure to Eq. (3.25.a) leads to the following first order equations

$$G_{\hat{a}}^{(1)} = \sum_{i=1}^{N} \Pi_i (h_{i1} + ih_{i2}) \hat{a} \tag{3.29.a}$$

$$F_{\hat{a}}^{(1)} \dot{F}_t^{(1)} = - \sum_{i=1}^{N} \Pi_i \frac{f_{1i}^*}{f_{1i}} (h_{i1} - ih_{i2}) e^{-2i\hat{\alpha}t} \hat{a}^* \tag{3.29.b}$$

where

$$\Pi_i = - \frac{\mu_i \ell_i^3 f_{1i} f_{1i}^* \omega_o^2 I_3}{4\hat{k}_{Ti} \hat{\alpha}} .$$

Integrating Eq. (3.29.b) with the condition (3.28), we obtain

$$F_{\hat{a}}^{(1)} F_t^{(1)} = \frac{1}{2i\hat{\alpha}} \sum_{i=1}^{N} \Pi_i \frac{f_{1i}^*}{f_{1i}} (h_{i1} - ih_{i2}) e^{-2i\hat{\alpha}t} \hat{a}^* . \tag{3.30}$$

Combination of Eq. (3.23.b) and Eq. (3.29.a) leads to the first order equation for a in the form

$$\dot{\hat{a}} = \epsilon^2 \sum_{i=1}^{N} \Pi_i (h_{i1} + ih_{i2}) \hat{a} . \tag{3.31}$$

Equation (3.31) is easily solved to obtain

$$\hat{a} = \hat{a}_o e^{\epsilon^2 \sum_{i=1}^{N} \Pi_i (h_{i1} + h_{i2}) t} \tag{3.32}$$

where $\hat{a}_o$ is an initial value of $\hat{a}$. Substituting this expression into Eqs. (3.19), we have the first order solutions for ω_1 and ω_2:

$$\omega_1 = 2\hat{\omega}_1 \hat{a}_o e^{\epsilon^2 \sum_{i=1}^{N} \Pi_i h_{i1} t} \ t \ \cos(\hat{\alpha} + \epsilon^2 \sum_{i=1}^{NN} \Pi_i h_{i2}) t$$

$$\omega_2 = 2\hat{\omega}_2 \hat{a}_o e^{\epsilon^2 \sum_{i=1}^{N} \Pi_i h_{i1} t} \sin(\hat{\alpha} + \epsilon^2 \sum_{i=1}^{N} \Pi_i h_{i2})\, t\,. \qquad (3.33)$$

Then, the formulae for the first ordre damping ratio δ_1^* and the first order frequency $\hat{\alpha}_1^*$ of nutational body motions are given as follows :

$$\delta_1^* = -\epsilon^2 \sum_{i=1}^{N} \Pi_i h_{i1} \qquad (3.34)$$

$$\hat{\alpha}_1^* = \hat{\alpha} + \epsilon^2 \sum_{i=1}^{N} \Pi_i h_{i2} \qquad (3.35)$$

It should be noted that the first order damping ratio (3.34) is the same as that obtained using the energy sink method (Appendix). Stability requires that the damping ratio be positive :

$$\delta_1 \geqq 0\,. \qquad (3.36)$$

The combination of the condition (3.20) and (3.36), as predicted by the energy sink method, requires that for the stability of the attitude motion the spin axis be the axis of maximum moment of inertia (the maximum inertia axis criterion).

We next proceed to the second order approximation. The equations for $G_{\hat{a}}^{(2)}$ and $F_{\hat{a}}^{(2)}$ $F_t^{(2)}$ can be found in a manner similar to the first order approximation. For the function $G_{\hat{a}}^{(2)}$, we obtain the equation

$$G_{\hat{a}}^{(2)} = \sum_{i,j=1}^{N} \Pi_i \Pi_j (\Gamma_{ij} h_i h_j - h_j g_i)\, \hat{a} \qquad (3.37)$$

where

$$\Gamma_{ij} = \frac{2i\hat{\alpha}}{(\omega_0^2 - \hat{\alpha}^2)} + \frac{i f_{1i}^* f_{1j}}{(2\hat{\alpha} f_{1i} f_{1j}^*)} + \frac{(f_{2i} f_{1j}^* - f_{2i}^* f_{1j})}{(f_{1i} f_{1j}^*)}$$

Hence, the second order equation for a is given by

$$\dot{a} = [\, \epsilon^2 \sum_{i=1}^{N} \Pi_i h_i + \epsilon^2 \sum_{i,j=1}^{N} \Pi_i \Pi_j \,(\Gamma_{ij} h_i h_j - h_j g_i)\,]\, \hat{a} \; . \tag{3.38}$$

We obtain from Eq. (3.38) the second order damping ratio δ_2^* for nutational body motions in the form

$$\delta_2^* = -\epsilon^2 \sum_{i=1}^{N} \Pi_i h_{1i} + \epsilon^4 \sum_{i,j=1}^{N} \Pi_i \Pi_j \, [\, \Gamma_{ij2} (h_{j1} h_{i2} + h_{j2} h_{i1}) - \Gamma_{ij1} (h_{j1} h_{i1} - h_{j2} h_{i2}) + (h_{j1} g_{i1} - h_{j2} g_{i2})\,] \tag{3.39}$$

where Γ_{ij1} is a real part of Γ_{ij} and Γ_{ij2} is an imaginery part of Γ_{ij}. The analytical stability criteria deduced from Eq. (3.39) give the result :

$$\left.\begin{aligned} &\delta_2^* \geqq 0 \ \ \text{implies stability} \\ &\delta_2^* < 0 \ \ \text{implies instability} \; . \end{aligned}\right\} \tag{3.40}$$

The condition (3.40) yields closed form stability criteria involving the properties of the flexible appendages such as the length and the mass of the appendages and the natural frequencies of the appendages.

As an example, a symmetrical spacecraft composed of a central rigid body and four equal appendages is investigated (Fig.3.2). First, the damping ratio and the frequency of the nutational body motion are calculated for the cases in which the system parameters are given as follows :

$$\text{Case 1 :} \quad \frac{I_3 - I}{I} = 0.020, \quad \frac{\mu \ell^3}{I} = 0.030$$

$$\frac{\delta}{k_{Ti}} = 0.1$$

Case 2 : $\dfrac{I_3 - I}{I} = 0.10, \quad \dfrac{\mu \ell^3}{I} = 0.14$

$$\frac{\delta}{k_{Ti}} = 0.1 .$$

Figures (3.3.a), (3.3.b) show the damping ratio calculated from the formulae (3.34), (3.39). In Figs. (3.4.a), (3.4.b) are plotted the curves of the nutational frequency by means of the formula (3.35). In the same figures, some numerical results obtained from Eqs. (3.15) by the eigenvalue analysis are shown for comparison. The first order damping ratio, which is the same as the result obtained by the energy sink method, is considerably different from the numerically computed solution. On the other hand, the second order damping ratio is rather close to the result obtained by the numerical computation. Next, the stability criterion (3.40) is checked for a number of critical cases against results obtained from Eqs. (3.15) by the digital computer eigenvalue analysis. Results presented in Fig. (3.5) show that good agreement is found.

3.4 Conclusions

On the basis of linearized equations of motion, the attitude motion of a freely spinning spacecraft with flexible appendages are discussed using an analytical method which utilizes the method of averaging. The damping ratio and the frequency of nutational body motions are obtained. It is shown that the damping ratio calculated as the first order solution of the present method is the same as the result obtained by the energy sink method. The second order damping ratio shows in good agreement with the numerically computed solution over a range of parameter values, for which the corresponding result obtained by the energy sink method is not accurate. Analytical stability criteria are deduced from the sign properties of the damping ratio. From the first order solution, the maximum inertia axis criterion, as predicted by the energy sink method, is obtained. From the second order solution, the stability condition involving the properties of flexible appendages is established. The results for this criterion are found to be in good agreement with the results obtained by the eigenvalue analysis.

APPENDIX

Here, we shall derive the damping ratio, Eq. (3.34), for nutational body motions on the basis of the energy sink method.

The first step in this procedure is the determination of the motion of the spacecraft under the assumption that the spacecraft is rigid. The equations of motion for such a system can be obtained by specializing Eqs. (3.15.a), (3.15.b) by setting $T_i = 0$:

$$\left.\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_0\omega_2 &= 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_0\omega_1 &= 0\,. \end{aligned}\right\} \qquad (1)$$

A solution to Eqs. (1) becomes

$$\left.\begin{aligned} \omega_1 &= -a\hat{\omega}_1 \sin\hat{\alpha}t \\ \omega_2 &= a\hat{\omega}_2 \operatorname{con}\hat{\alpha}t \end{aligned}\right\} \qquad (2)$$

where

$$\hat{\omega}_1 = \frac{1}{I_1}\left(\frac{I_3}{I_2} - 1\right)^{\frac{1}{2}}$$

$$\hat{\omega}_2 = \frac{1}{I_2}\left(\frac{I_3}{I_1} - 1\right)^{\frac{1}{2}}$$

$$\hat{\alpha} = \left[\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right)\right]^{\frac{1}{2}} \omega_o .$$

The second step is to determine the motion of the appendages under the assumption that the motion of the appendages has a negligible effect on the motion of the spacecraft. Equations which govern the motions of the appendages can be developed by substituting Eqs. (2) into Eqs. (3.15.c)

$$\begin{aligned} \ddot{T}_i + 2\delta_i \dot{T}_i + k_{Ti}^2 T_i = {} & C\gamma_i \epsilon(\omega_o \hat{\omega}_1 - \hat{\omega}_2 \hat{\alpha})\, a \sin \hat{\alpha} t \\ & - S\gamma_i \epsilon\, (\omega_o \hat{\omega}_2 - \hat{\omega}_1 \hat{\alpha})\, a \cos \hat{\alpha} t . \end{aligned} \tag{3}$$

$$(i=1,\ldots\ldots N)$$

A steady state solution of Eq. (3) is found to be

$$\begin{aligned} T_i = {} & \frac{C\gamma_i \epsilon a}{2\hat{k}_{Ti}} (\omega_o \hat{\omega}_1 - \hat{\omega}_2 \hat{\alpha})\,(h_{i2} \sin \hat{\alpha} t - h_{i1} \cos \hat{\alpha} t) \\ & - \frac{S\gamma_i \epsilon a}{2\hat{k}_{Ti}} (\omega_o \hat{\omega}_2 - \hat{\omega}_1 \hat{\alpha})\,(h_{i2} \cos \hat{\alpha} t + h_{i1} \sin \hat{\alpha} t) \end{aligned} \tag{4}$$

where

$$h_{i1} = \frac{\delta_i}{\delta_i^2 + (\hat{k}_{Ti} - \hat{\alpha})^2} - \frac{\delta_i}{\delta_i^2 + (\hat{k}_{Ti} + \hat{\alpha})^2}$$

$$h_{i2} = \frac{(\hat{k}_{Ti} - \hat{\alpha})}{\delta_i^2 + (\hat{k}_{Ti} - \hat{\alpha})^2} + \frac{(\hat{k}_{Ti} + \hat{\alpha})}{\delta_i^2 + (\hat{k}_{Ti} + \hat{\alpha})^2}$$

$$\hat{k}_{Ti} = (k_{Ti}^2 - \delta_i^2)^{\frac{1}{2}}$$

The third step in the procedure is to compute the rate at which the energy is dissipated in the appendages. The energy E dissipated per cycle is given by

$$E = 2 \sum_{i=1}^{N} \mu_i \ell_i^3 \delta_i \int_o^{\frac{2\pi}{\hat{\alpha}}} \dot{T}_i^2 \, dt \tag{5}$$

Substituting Eq. (4) into Eq. (5), we find

$$E = \frac{\epsilon^2 \pi}{2} \sum_{i=1}^{N} \frac{\mu_i \ell i^3 h_{i1}}{\hat{k}_{Ti}} \Big\{ [C\gamma_i(\omega_o \hat{\omega}_1 - \hat{\alpha}\hat{\omega}_2)]^2 \tag{6}$$

(6)

$$+ [S\gamma_i(\omega_o \hat{\omega}_2 - \hat{\alpha}\hat{\omega}_1)]^2 \Big\} a^2 .$$

The average rate of the energy dissipation K (which is defined by $K=-E/\frac{2\pi}{\hat{\alpha}}$) is given (by the use of Eq. (6))

$$K = -\epsilon^2 \hat{\alpha} \sum_{i=1}^{N} \frac{\mu_i \ell_i^3 h_{i1}}{4\hat{k}_{Ti}} \Big[C^2 \gamma_i(\omega_o \hat{\omega}_1 - \hat{\alpha}\hat{\omega}_2)^2 + \tag{7}$$

$$S^2 \gamma_i(\omega_o \hat{\omega}_2 - \hat{\alpha}\hat{\omega}_1)^2 \Big] a^2 .$$

The final step is to relate the energy disipation rate to changes in the attitude motion. This is done as follows : the spacecraft is treated as rigid, in which case the kinetic energy for this idealized system is

$$2T = I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \tag{8}$$

and the square of the system angular momentum about its center of mass is

$$\ell^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 . \tag{9}$$

Differentiating Eqs. (8), (9) with respect to time, we obtain

$$\left.\begin{aligned} \dot{T} &= I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3 \\ 0 &= I_1^2 \omega_1 \dot{\omega}_1 + I_2^2 \omega_2 \dot{\omega}_2 + I_3^2 \omega_3 \dot{\omega}_3 \end{aligned}\right\} \tag{10}$$

Elimination of $\omega_3 \dot{\omega}_3$ between Eqs. (10) leads to

$$\dot{T} = I_1 (1 - \frac{I_1}{I_3}) \omega_1 \dot{\omega}_1 + I_2 (1 - \frac{I_2}{I_3}) \omega_2 \dot{\omega}_2 . \tag{11}$$

If the effect of the energy disipation mechanism is taken into account, we can assume that the amplitude of the nutational body motion, a, is a slowly varying function of time. Then, we find, from Eqs. (2), the time derivatives of ω_1 and ω_2 as follows :

$$\left.\begin{aligned} \dot{\omega}_1 &= -\dot{a} \hat{\omega}_1 \sin \hat{\alpha} t - a \hat{\omega}_1 \hat{\alpha} \cos \hat{\alpha} t \\ \omega_2 &= \dot{a} \hat{\omega}_2 \cos \hat{\alpha} t - a \hat{\omega}_2 \hat{\alpha} \sin \hat{\alpha} t \quad . \end{aligned}\right\} \tag{12}$$

Substitution of Eqs. (12) into Eq. (11) leads to

$$\begin{aligned} \dot{T} = I_1 (1 - \frac{I_1}{I_3}) (\dot{a} \hat{\omega}_1 \sin \hat{\alpha} t + a \hat{\omega}_1 \hat{\alpha} \cos \hat{\alpha} t) a \hat{\omega}_1 \sin \hat{\alpha} t \\ + I_2 (1 - \frac{I_2}{I_3}) (\dot{a} \hat{\omega}_2 \cos \hat{\alpha} t - a \hat{\omega}_2 \hat{\alpha} \sin \hat{\alpha} t) a \hat{\omega}_2 \cos \hat{\alpha} t \end{aligned} \tag{13}$$

Averaging over the period $\frac{2\pi}{\hat{\alpha}}$, we obtain the rate of the change of the system kinetic energy :

$$\bar{\dot{T}} = \frac{(I_1 I_2)^2}{I_3} \hat{\omega}_1^2 \hat{\omega}_2^2 a \dot{a} . \tag{14}$$

The crux of the energy sink method involves equating $\bar{\dot{T}}$ to K of Eq. (7), thus leading to a differential equation for a,

$$\dot{a} = -\frac{\epsilon^2 I_3 \omega_o^2}{\hat{\alpha}} \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{4\hat{k}_{Ti}} \Big[C^2 \gamma_i (\omega_o \hat{\omega}_1 - \hat{\alpha}\hat{\omega}_2)^2 + \tag{15}$$

$$S^2 \gamma_i (\omega_o \hat{\omega}_2 - \hat{\alpha}\hat{\omega}_1)^2 \Big] a .$$

The damping ratio for nutational body motions is given by

$$\delta^* = \frac{\epsilon^2 I_3 \omega_o^2}{\hat{\alpha}} \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{4\hat{k}_{Ti}} \Big[C^2 \gamma_i (\omega_o \hat{\omega}_1 - \hat{\alpha}\hat{\omega}_2)^2 \tag{16}$$

$$+ S^2 \gamma_i (\omega_o \hat{\omega}_2 - \hat{\alpha}\hat{\omega}_1)^2 \Big] .$$

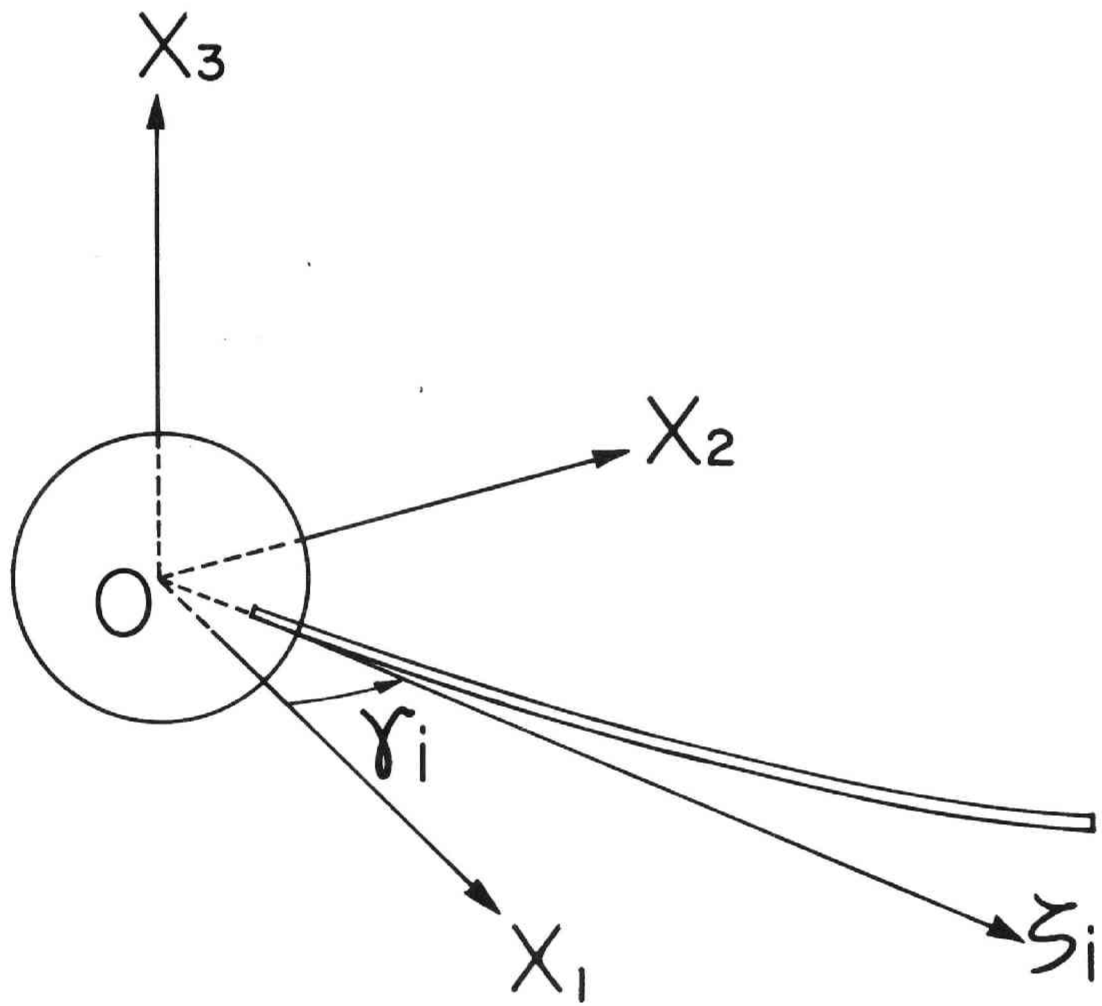

Fig. 3.1 Spacecraft configuration

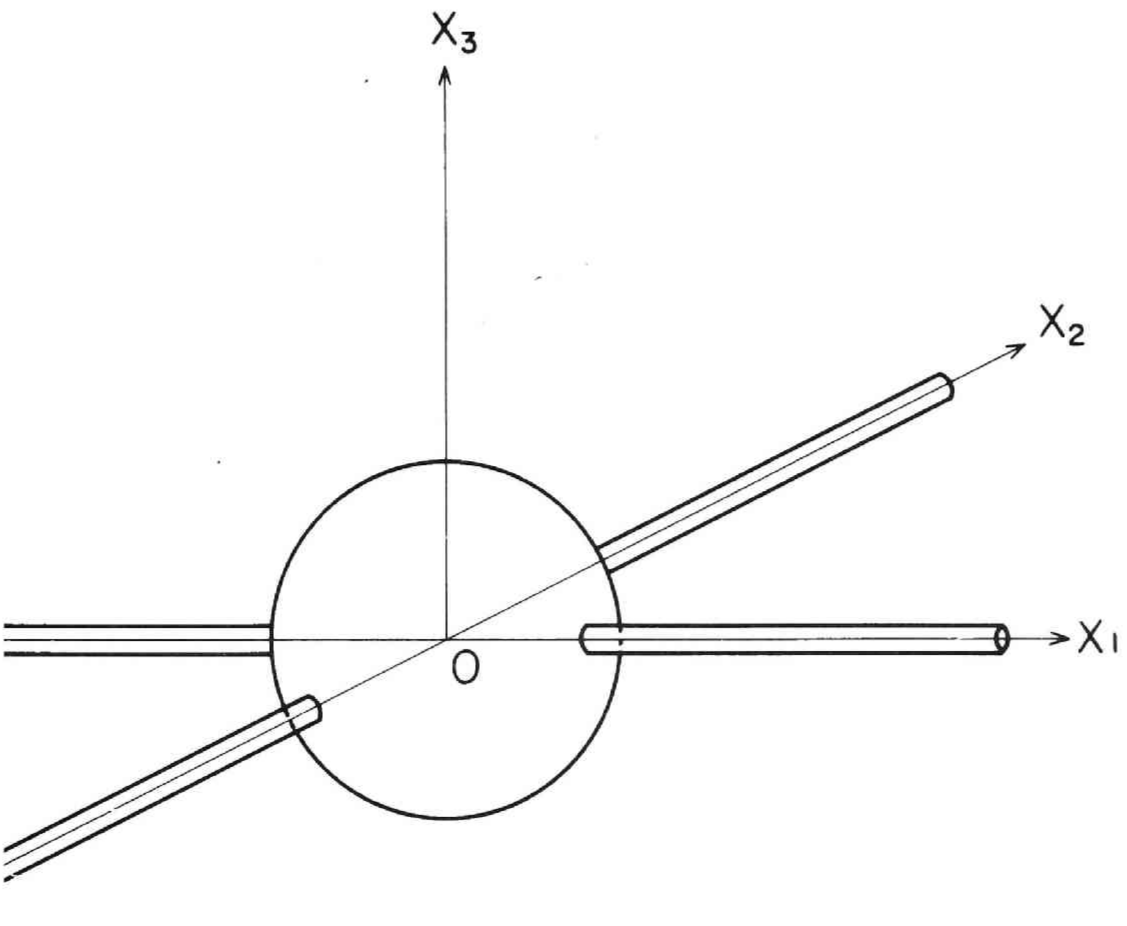

Fig. 3.2 Symmetrical spacecraft configuration

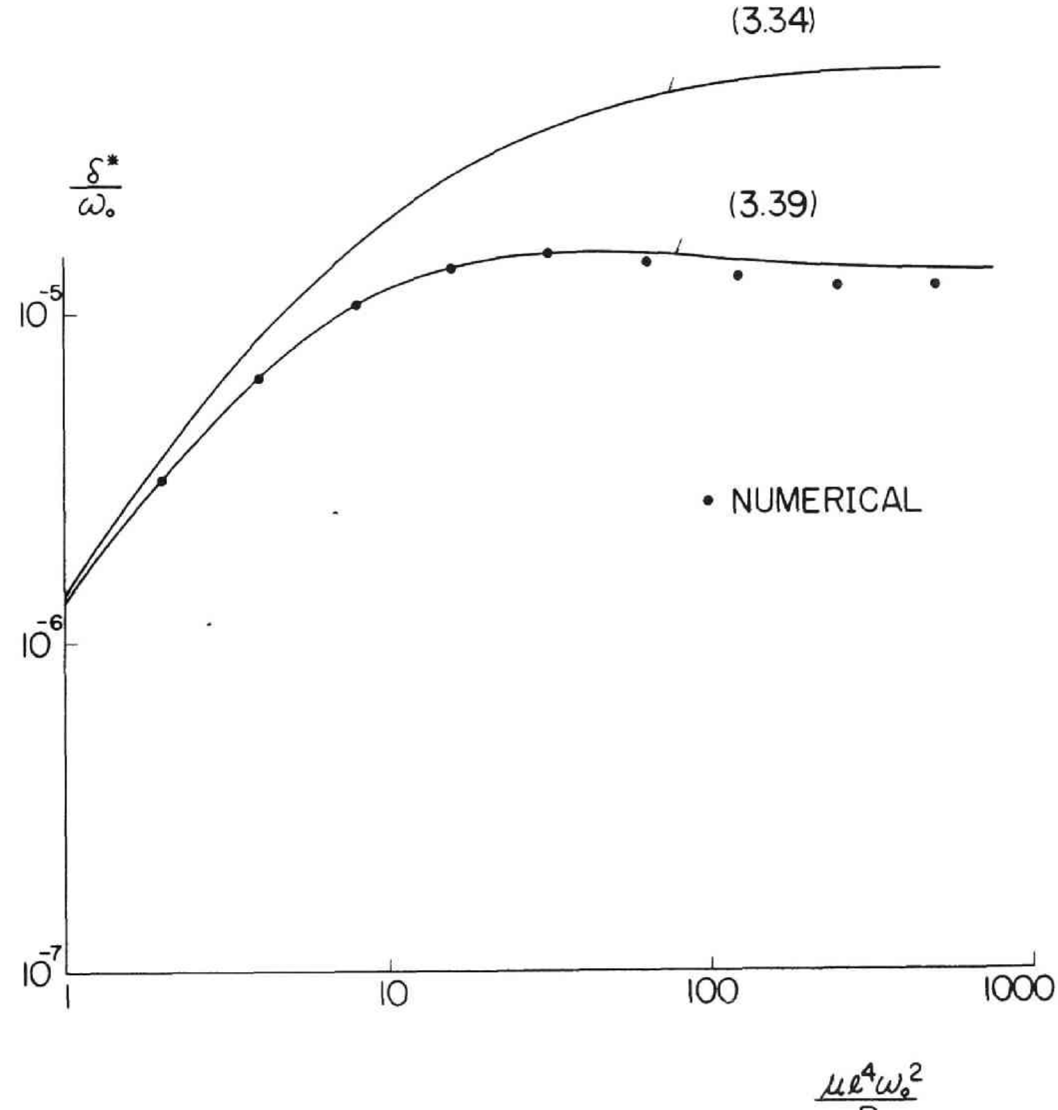

Fig. 3.3.a Damping ratio for nutational body motion (case 1)

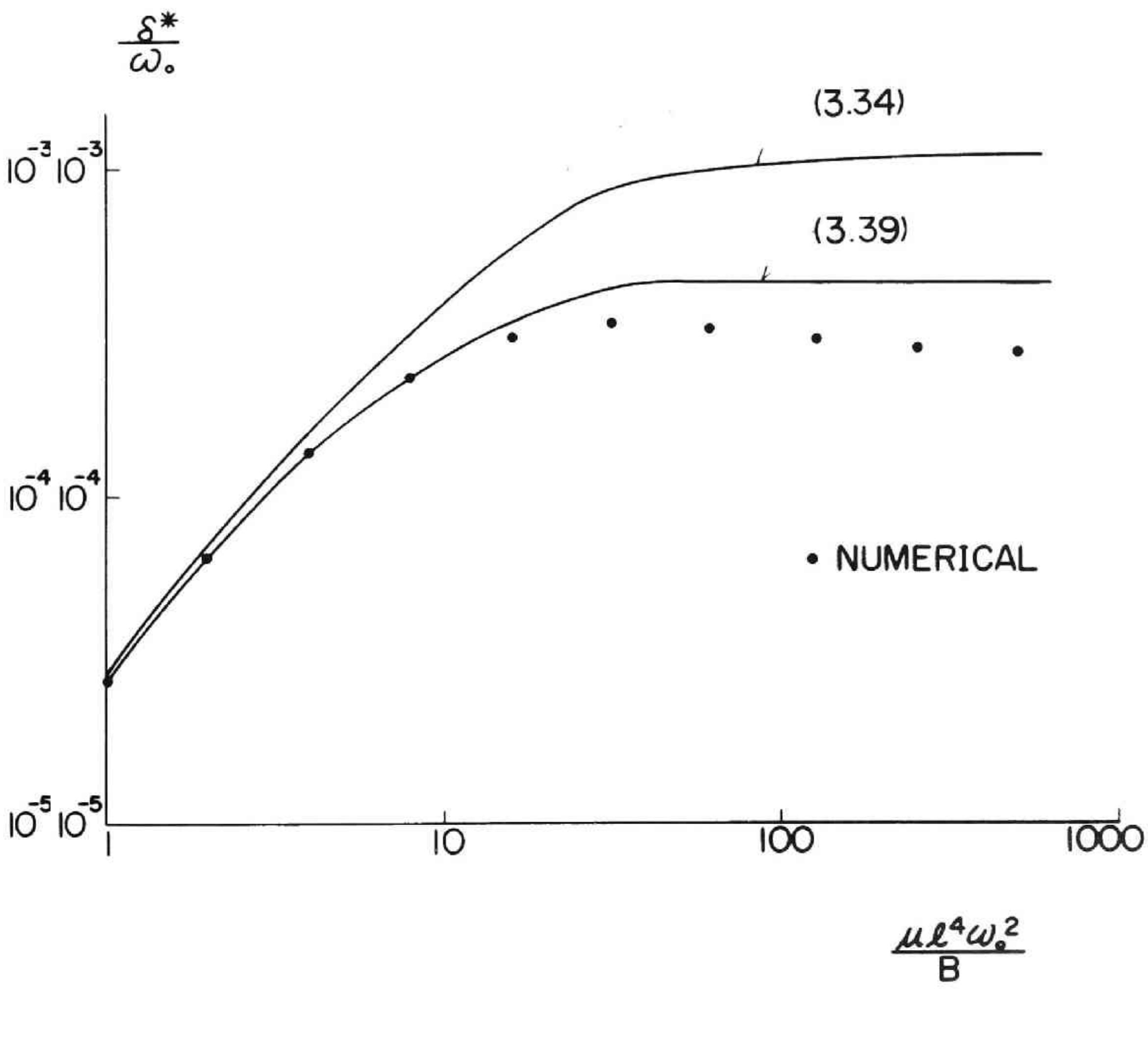

Fig. 3.3.b Damping ratio for nutational body motion (case 2)

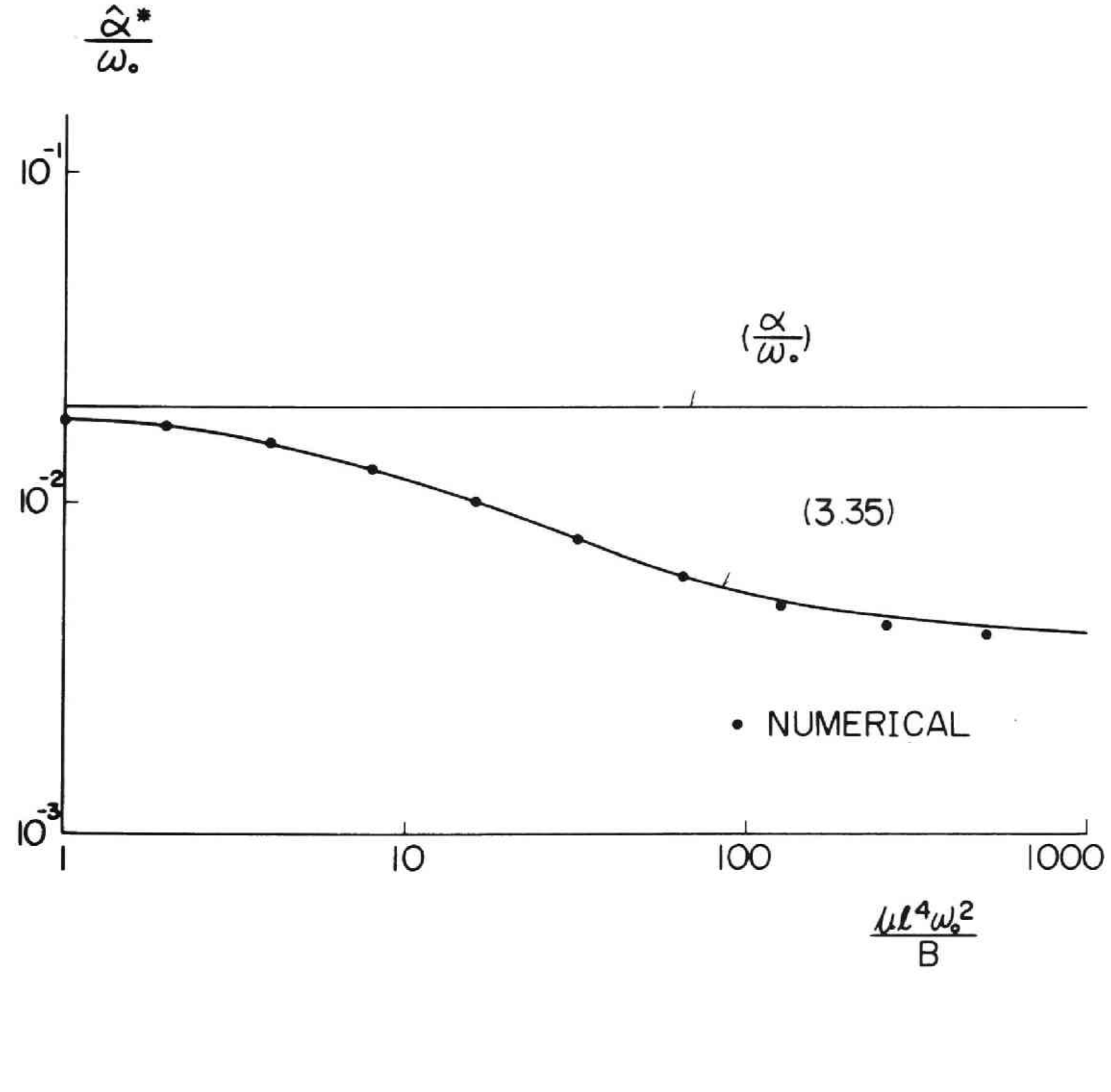

Fig. 3.4.a Frequency of nutational body motion (case 1)

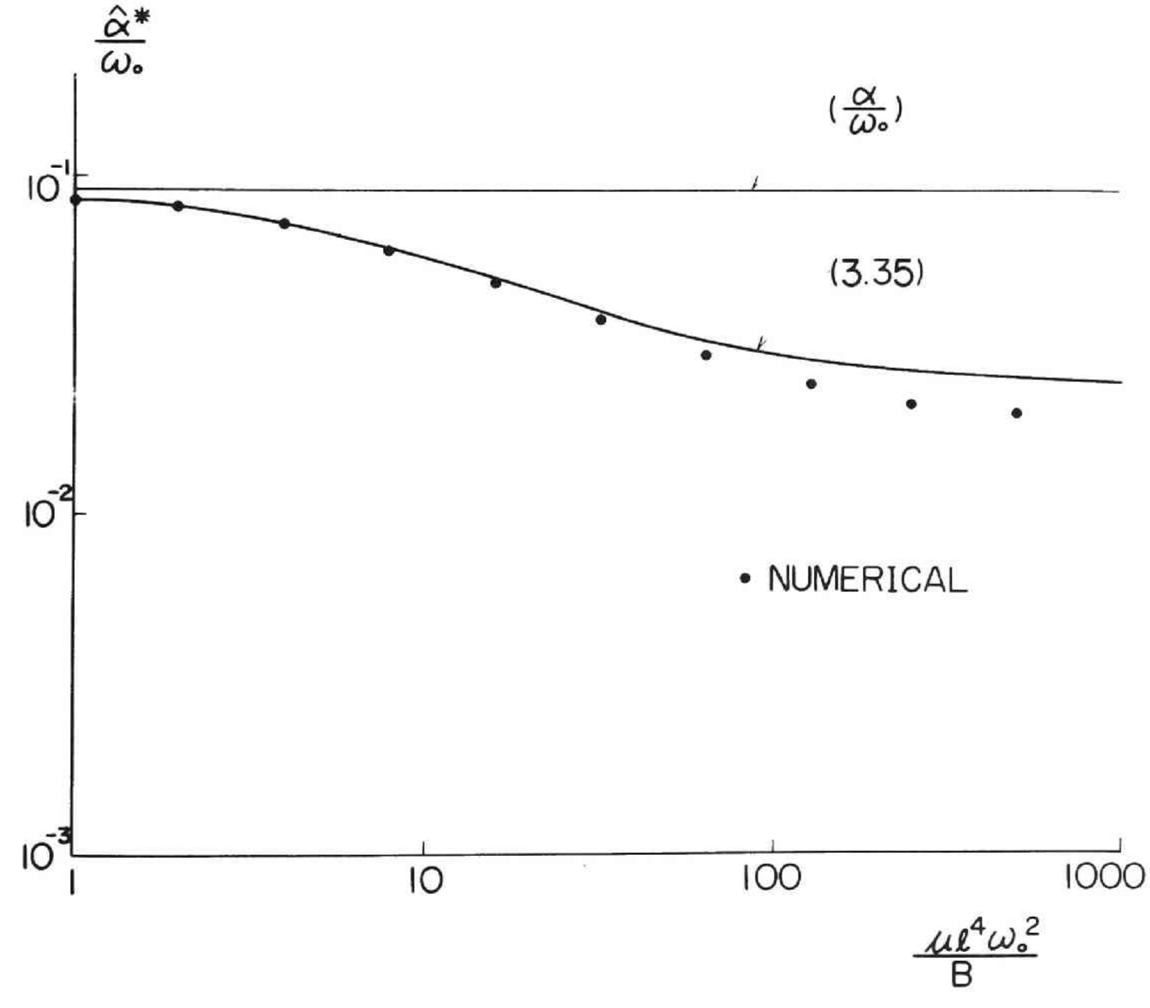

Fig. 3.4.b Frequency of nutational body motion (case 2)

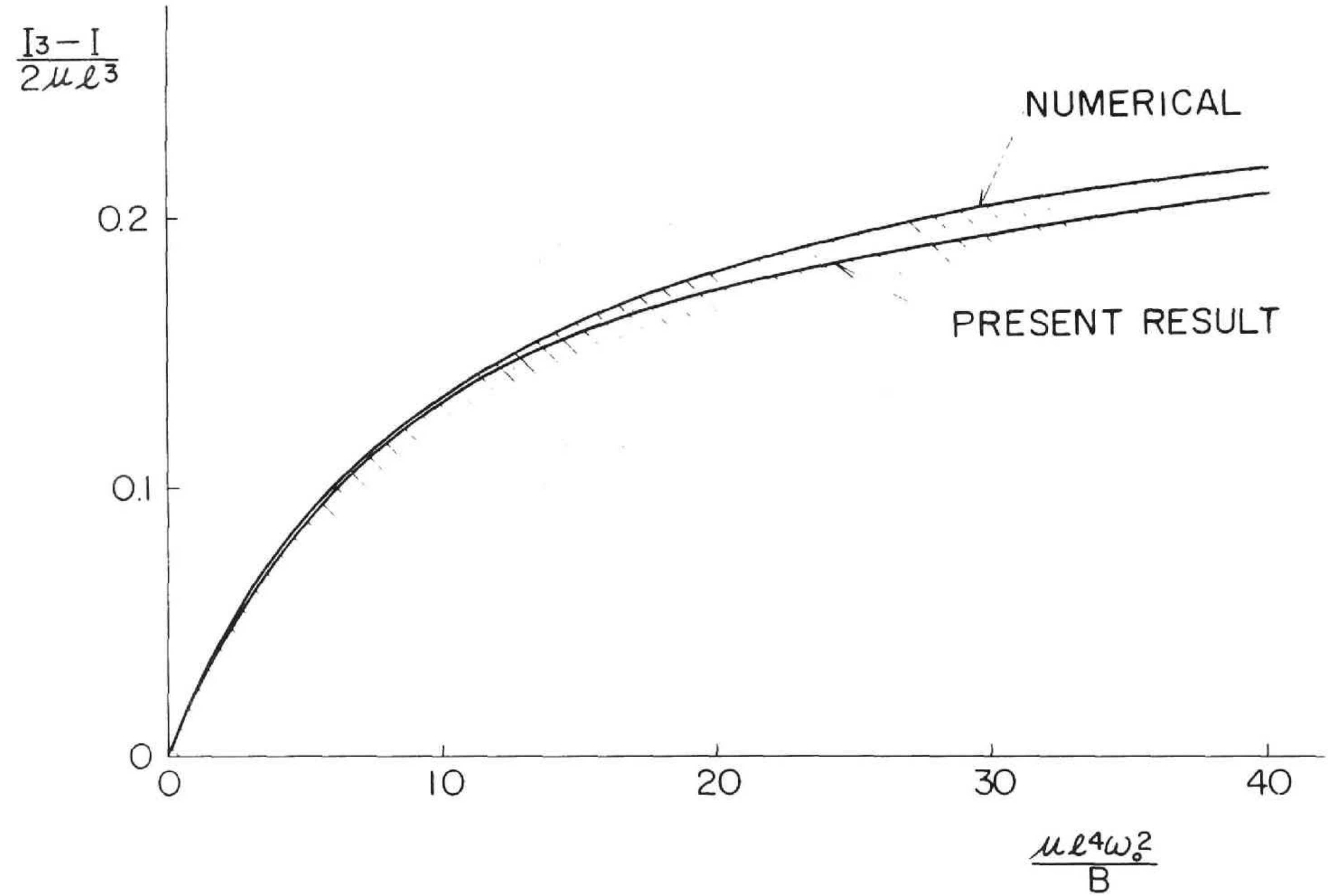

Fig. 3.5 Stability regions (hatched regions are unstable)

CHAPTER IV

NUTATION DAMPING OF A SPINNING SPACECRAFT HAVING FLEXIBLE APPENDAGES DUE TO A NONLINEAR INTERNAL RESONANCE

4.1 Introduction

The attitude of a spacecraft spinning in the absence of external forces is not a constant when energy dissipation takes place. Elastic vibrations of flexible appendages, induced by gyroscopic action, also result in a disipation of energy and a change in the attitude of a spacecraft. There have been many studies of the effect of vibrations of appendages on the attitude motion of a spacecraft, but these studies were confined to the case where appendages are excited at nonresonance conditions.

Interesting phenomena take place when appendages are excited at near resonance conditions. In this chapter, we shall investigate heavy damping characteristics of nutational body motions of a freely spinning spacecraft with flexible appendages due to a certain nonlinear internal resonance between vibrations of the appendages and nutational body motions. A spinning spacecraft composed of a central rigid body and flexible appendages normal to the spin axis is considered. Vibrations of the flexible appendages perpendicular to the spin axis are induced by the nutational body motion through the second order terms of the angular velocity of the spacecraft. The appendage vibrations, which are usually small, build up to larger values when the frequencies of the appendages are nearly equal to twice the nutational frequency of the spacecraft. As a result, a large energy transfer takes place between the appendage vibrations and the nutational body motion, and the disipation of energy derived from it results in a large damping of nutational body motions. The method of averaging is used to obtain an analytical expression for the damping of the nutational body motion and the accuracy of this expression is confirmed by digital computer simulations.

4.2 Equations of motion

Consider a symmetrical spacecraft composed of a heavy rigid central body and four equal mutually-orthogonal flexible appendages, as shown in Fig. 4.1. Let the reference axes (X_1, X_2, X_3) be assigned to the spacecraft in such a way that the X_3 axis coincides with the spin axis, and the X_1 and X_2 axes are coincident with the appendages when the appendages are undeformed.

For an appendage i, an axis system (ζ_i, η_i, ξ_i) is defined such that ξ_i axis coincides with the spin axis, and the ζ_i axis coincides with the appendage i in the undeformed state. Hence, if we denote an angle between the axes X_1 and ζ_i by γ_i, it follows that $\gamma_i = \frac{\pi}{2}(i-1)$ $(i=1,\ldots,4)$. Let the angular velocity of the (X_1, X_2, X_3) axes be given by (ω_1, ω_2, ω_3) in the (X_1, X_2, X_3) reference frame.

The mass center of the total configuration is assumed to be fixed at the origin of the axis system (X_1, X_2, X_3). When the vibrations of the appendages in a spin plane are excited at near resonance conditions, the out of plane vibrations of the appendages are excited at nonresonance conditions. However, since vibrations of appendages perpendicular to the spin plane do not have the essential influence on the phenomena discussed here, we neglect, in this analysis, the vibrations of the appendages perpendicular to the spin plane. Then, the total kinetic energy

T can be written in the form

$$2T = I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 + \sum_{i=1}^{4} \mu \int_0^1 [\dot{u}_i^2 + 2\omega_3\zeta_i\dot{u}_i$$

$$+ \omega_3^2 u_i^2 - 2\omega_1\omega_2 C2\gamma_i\zeta_i u_i]\, ds_i \tag{4.1}$$

where I and I_3 are the moments of inertia of the total system about X_1 (or X_2), X_3 axes, respectively, u_i an elastic deformation of an appendage i in the spin plane, μ the mass per unit length of the appendages, ℓ the length of the appendages, ds_i the arc length along the appendage i, $C2\gamma_i = \cos 2\gamma_i$. We shall consider here the change in the attitude of the spacecraft undergoing small deformations of the appendages. Hence, Equations of motion can be linearized in the deformation coordinates for the appendages, but the angular velocity components ponents of the spacecraft are unrestricted. The kinetic energy expression, then becomes

$$2T = I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 + \sum_{i=1}^{4} \mu \int_0^1 [\dot{u}_i^2 + 2\omega_3\zeta_i\dot{u}_i$$

$$- 2\omega_1\omega_2 C2\gamma_i\zeta_i u_i + \omega_3^2 (u_i^2 - \frac{1}{2}(\ell^2 - \zeta_i^2)(\frac{\partial u_i}{\partial \zeta_i})^2]\, d\zeta_i\,. \tag{4.2}$$

We shall ignore external forces in this analysis. The potential energy U then consists entirely of the elastic strain energy of the appendages, i.e.,

$$2U = \sum_{i=1}^{4} B \int_0^1 (\frac{\partial^2 u_i}{\partial \zeta_i^2})^2 d\zeta_i \tag{4.3}$$

where B is the bending stiffness of the appendages.

The energy dissipation which results from elastic deformations of the appendages is represented by Rayleigh's dissipation function F, which is given by

$$F = \sum_{i=1}^{4} \mu\delta \int_0^{\ell} \dot{u}_i^2 d\zeta_i \tag{4.4}$$

where δ is the damping ratio of the appendages.

We expand the deflections u_i in terms of functions $E_n(\hat{\zeta})$

$$u_i(\zeta_i, t) = \ell \sum_{n=1}^{\infty} P_{in}(t) E_n(\hat{\zeta}) \tag{4.5}$$

where $P_{in}(t)$ are modal deformation coordinates which are functions of time. The functions $E_n(\hat{\zeta})$ are normal modes associated with a cantilever, which satisfy the following boundary value problems

$$\left.\begin{aligned} &\frac{d^4 E_n}{d\hat{\zeta}^4}(\hat{\zeta}) - \lambda_n{}^4 E_n(\hat{\zeta}) = 0 \\ &\hat{\zeta} = 0, \quad E_n(\hat{\zeta}) = 0, \quad \frac{dE_n(\hat{\zeta})}{d\hat{\zeta}} = 0 \\ &\hat{\zeta} = 1, \quad \frac{d^2 En(\hat{\zeta})}{d\hat{\zeta}^2} = 0, \quad \frac{d^3 E_n(\hat{\zeta})}{d\hat{\zeta}^3} = 0 \end{aligned}\right\} \tag{4.6}$$

where λ_n^4 are the eigenvalues of the normal modes $E_n(\hat{\zeta})$ and $\hat{\zeta} = \frac{\zeta_i}{\ell}$. In addition, they are normalized such that

$$\int_0^1 E_n(\hat{\zeta}) E_m(\hat{\zeta}) d\hat{\zeta} = \delta_{n,m}$$

where $\delta_{n,m}$ is Kronecker's delta. In the following, the appendages are assumed to be excited near or below the first natural frequency, so that a reasonable approximation to the dynamical behavior of the spacecraft is provided by a reduced system of equations of motion obtained by truncating the series at the first mode, i.e., n=1. Equations (4.2), (4.3), (4.4), on substituting Eq. (4.5), truncating the series at n=1 and neglecting the suffix 1, become

$$2T = I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 + \sum_{i=1}^{N} \mu\ell^3 [\dot{P}_i^2 + 2\omega_3 \epsilon \dot{P}_i$$

$$-2\omega_1\omega_2 C2\gamma_i \epsilon P_i + \omega_3^2 (1-\beta) P_i^2] \tag{4.7}$$

$$2U = \sum_{i=1}^{4} \frac{B\lambda^4}{\ell} P_i^2 \tag{4.8}$$

$$F = \sum_{i=1}^{4} \mu\ell^3 \delta \dot{P}_i^2 \tag{4.9}$$

where

$$\epsilon = \int_o^1 \hat{\xi} E(\hat{\xi}) d\hat{\xi} = -0.5688$$

$$\beta = \frac{1}{2} \int_o^1 (1 - \hat{\xi}^2)\left(\frac{dE(\hat{\xi})}{d\hat{\xi}}\right)^2 d\hat{\xi} = 1.193$$

Since the coordinates $P_i(t)$ are generalized coordinates, the Lagrange equations for them take the forms (32)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{P}_i}\right) + \left(\frac{\partial F}{\partial \dot{P}_i}\right) - \left(\frac{\partial L}{\partial P_i}\right) = 0 \tag{4.10}$$

where the Lagrangian L is given by

$$L = T - U.$$

On the other hand, since the coordinates ω_1, ω_2, ω_3 are so called quasi-coordinates, the corresponding equations for them take the forms (32)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + \omega_2\left(\frac{\partial T}{\partial \omega_3}\right) - \omega_3\left(\frac{\partial T}{\partial \omega_2}\right) = N_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) + \omega_3\left(\frac{\partial T}{\partial \omega_1}\right) - \omega_1\left(\frac{\partial T}{\partial \omega_3}\right) = N_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) + \omega_1\left(\frac{\partial T}{\partial \omega_2}\right) - \omega_2\left(\frac{\partial T}{\partial \omega_1}\right) = N_3 \qquad (4.11)$$

where N_1, N_2, N_3 are external torque components about X_1, X_2, X_3 axes, respectively. Since external torques are ignored here, it follows that

$$N_1 = N_2 = N_3 = 0\,. \qquad (4.12)$$

Substituting Eqs. (4.7), (4.8), (4.9), (4.12) into Eqs. (4.10), (4.11), we find the equations of motion as follows :

$$I\dot{\omega}_1 + (I_3 - 1)\omega_3\omega_2 + \epsilon\mu\ell^3 \sum_{i=1}^{4} [- C2\gamma_i(\dot{P}_i\omega_2 + P_i\dot{\omega}_2)$$

$$+ \omega_2\dot{P}_i + C2\gamma_i\omega_1\omega_3 P_i] = 0 \qquad (4.13.a)$$

$$I\dot{\omega}_2 - (I_3 - 1)\omega_3\omega_1 + \epsilon\mu\ell^3 \sum_{i=1}^{4} [-C2\gamma_i(\dot{P}_i\omega_1 + P_i\dot{\omega}_1)$$

$$-\omega_1\dot{P}_i - C2\gamma_i\omega_2\omega_3 P_i] = 0 \qquad (4.13.b)$$

$$I_3\dot{\omega}_3 + \epsilon\mu\ell^3 \sum_{i=1}^{4} \ddot{P}_i = 0 \qquad (4.13.c)$$

$$\ddot{P}_i + 2\delta\dot{P}_i + [\frac{\lambda^4 B}{\mu\ell^4} + (\beta-1)\omega_3^2] P_i + \epsilon C2\gamma_i\omega_1\omega_2 = 0\,.$$

$$(i=1,\ldots\ldots, 4) \qquad (4.13.d)$$

The attitude motion of this class of spacecraft may be classified into two modes : the first mode, which describes deflections of the appendages induced by nutational body motions, and the second mode, which describes nutational body motions, induced by deflections of the appendages. Since the second mode has no practical importance, only the first mode of the motion is investigated in the present analysis. Hence, without loss of generality, the initial conditions are taken to be

$$\left.\begin{array}{lll} \omega_1 = \hat{\omega}, & \omega_2 = 0, & \omega_3 = \omega_o \\ P_i = 0, & \dot{P}_i = 0 & \text{at } t = 0. \end{array}\right\} \tag{4.14}$$

4.3 Analysis

From Eqs. (4.13) and the initial conditions (4.14), it can be concluded that

$$\left.\begin{array}{l} \sum_{i=1}^{4} P_i = 0 \\ \omega_3 = \omega_o . \end{array}\right\} \tag{4.15}$$

Equations of motion, on substitution of Eqs. (4.15) and introduction of a new variable

$$P = \sum_{i=1}^{4} C2\gamma_i P_i \tag{4.16}$$

become

$$I\dot{\omega}_1 + (I_3 - I)\omega_o \omega_2 = \epsilon\mu\ell^3 (\dot{P}\omega_2 + P\dot{\omega}_2 - \omega_o \omega_1 P) \tag{4.17.a}$$

$$I\dot{\omega}_2 - (I_3 - I)\omega_o \omega_1 = \epsilon\mu\ell^3 (\dot{P}\omega_1 + P\dot{\omega}_1 + \omega_o \omega_2 P) \tag{4.17.b}$$

$$\ddot{P} + 2\delta\dot{P} + k_p^2 P = -4\epsilon\omega_1 \omega_2 \tag{4.17.c}$$

where

$$k_p^2 = \frac{\lambda^4 B}{\mu\ell^4} + (\beta - 1)\omega_o^2 .$$

First, let us treat these equations on the basis of the method of analysis devised in chapter III. We write the variables ω_1, ω_2 in the forms

$$\left.\begin{aligned}\omega_1 &= \hat{\omega}\,(ae^{i\hat{\alpha}t} + a^* e^{-i\hat{\alpha}t}) \\ \omega_2 &= -i\hat{\omega}\,(ae^{i\hat{\alpha}t} - a^* e^{-i\hat{\alpha}t})\end{aligned}\right\} \tag{4.18}$$

where a^* is a complex conjugate of a,

$$\hat{\alpha} = (\frac{I_3}{I} - 1)\,\omega_o\,.$$

Substitution of Eqs. (4.18) into Eqs. (4.17) leads to

$$\dot{a} = (\frac{\epsilon\mu\ell^3}{I\hat{\omega}})\,[\,i\hat{\omega}\,a^*\dot{P} + i\hat{\omega}\dot{a}^* P + (\,\hat{\alpha} - \omega_o)\hat{\omega}a^* P\,]\,e^{-2i\hat{\alpha}t} \tag{4.19.a}$$

$$\ddot{P} + 2\delta\dot{P} + k_p^{\,2} P = 4i\epsilon\hat{\omega}^2\,(a^2 e^{2i\hat{\alpha}t} - a^* e^{-2i\hat{\alpha}t})\,. \tag{4.19.b}$$

Integrating Eq. (4.19.b), we find

$$\begin{aligned} P &= \frac{\epsilon}{2i\hat{k}_p}\int^t 4i\hat{\omega}^2\,(a^2 e^{2i\hat{\alpha}t} - a^{*2} e^{-2i\hat{\alpha}t})e^{(i\hat{k}_p - \delta)(t-t')}\,dt' \\ &\quad + \text{complex conjugate part} \\ &= \epsilon L_{ir}\,(a, a^*, t) \end{aligned} \tag{4.20}$$

where

$$\hat{k}_p = (k_p^{\,2} - \delta^2)^{\frac{1}{2}}$$

Substituting this expression for P into Eq. (4.19.a), we obtain the following integro-differential equation

$$\begin{aligned}\dot{a} = (\frac{\epsilon^2\mu\ell^3}{I\hat{\omega}})\Big\{ & i\hat{\omega}a^*\,\frac{d}{dt}\,L_{ir}\,(a, a^*, t) \\ & + [\,i\hat{\omega}\dot{a}^* + (\hat{\alpha} - \omega_o)\,\hat{\omega}a^*\,]\,L_{ir}\,(a, a^*, t) \quad e^{-2i\hat{\alpha}t}\end{aligned} \tag{4.21}$$

Let us expand the variable a in the form

$$\left.\begin{aligned} a &= \hat{a} + \sum_{n=1}^{\infty} \epsilon^{2n} F_{\hat{a}}^{(n)} (\hat{a}, \hat{a}^*) F_t^{(n)} (t) \\ \dot{\hat{a}} &= \sum_{n=1}^{\infty} \epsilon^{2n} G_{\hat{a}}^{(n)} (\hat{a}, \hat{a}^*) . \end{aligned}\right\} \tag{4.22}$$

On substituting Eq. (4.22) and integrating by parts, the function L_{ir} (a, a^*, t) is expressed as the power series of ϵ^2 in the form

$$L_{ir}(a, a^*, t) = \left(\frac{2\hat{\omega}^2}{\hat{k}_p}\right) [\, h\hat{a}^2 e^{i2\hat{\alpha}t} + h^* \hat{a}^{*2} e^{-i2\hat{\alpha}t} \,] + O(\epsilon^2) \tag{4.23}$$

where

$$h = h_1 + ih_2$$

$$h_1 = \frac{\delta}{\delta^2 + (2\hat{\alpha} - \hat{k}_p)^2} - \frac{\delta}{\delta^2 + (2\hat{\alpha} + \hat{k}_p)^2}$$

$$h_2 = \frac{(\hat{k}_p - 2\hat{\alpha})}{\delta^2 + (2\hat{\alpha} - \hat{k}_p)^2} + \frac{(\hat{k}_p + 2\hat{\alpha})}{\delta^2 + (2\hat{\alpha} + \hat{k}_p)^2} .$$

Substituting this and Eqs. (4.22) into Eq. (4.21) and equating like powers of ϵ^2, we obtain a series of equations as follows :

$$G_{\hat{a}}^{(1)} + F_{\hat{a}}^{(1)} \dot{F}_t^{(1)} = \left(\frac{2\mu\ell^3 \hat{\omega}^2}{I\hat{k}_p}\right) [\, -(\hat{\alpha} + \omega_o)\, \hat{a}^* \hat{a}^2 h + (3\hat{\alpha} - \omega_o)\, h^* \hat{a}^{*3} e^{-4i\hat{\alpha}t} \,] \tag{4.24.a}$$

$$\cdots$$

Application of a similar procedure devised in chapter III to Eq. (4.24.a) leads to the first order equations for a

$$
\left.\begin{aligned}
a &= \hat{a} \\
\dot{\hat{a}} &= -\left(\frac{2\epsilon^2 \mu \ell^3 \hat{\omega}^2}{I\hat{k}_p}\right)(\hat{\alpha}+\omega_o) h \hat{a}^* \hat{a}^2 .
\end{aligned}\right\} \qquad (4.25)
$$

Let us seek a solution in the form

$$
\hat{a} = re^{i \textcircled{H}} \qquad (4.26)
$$

From Eqs. (4.18), the quantity r is found to be proportional to the amplitudes of the angular velocity components ω_1, ω_2. Furthermore, the quantity r is proportional to the amplitude of the nutational body motion, $\sqrt{\omega_1^2 + \omega_2^2}/\omega_o$. Substituting Eq. (4.26) in Eqs. (4.25), we obtain for r and Ⓗ the equations

$$
\dot{r} = -\left(\frac{2\epsilon^2 \mu \ell^3 \hat{\omega}^2}{I\hat{k}_p}\right)(\hat{\alpha}+\omega_o)\, h_1 r^3 \qquad (4.27.a)
$$

$$
\dot{\textcircled{H}} = -\left(\frac{2\epsilon^2 \mu \ell^3 \hat{\omega}^2}{I\hat{k}_p}\right)(\hat{\alpha}+\omega_o)\, h_2 r^2 . \qquad (4.27.b)
$$

On the other hand, the initial conditions (4.14) become

$$
t = 0, \qquad r = \frac{1}{2}, \qquad \textcircled{H} = 0 . \qquad (4.27.c)
$$

Solving Eq. (4.27.a) with the conditions (4.27.c), we have, in a straightforward manner, the first order approximation of r in the form

$$
r = \frac{1}{\left[4 + \dfrac{4\epsilon^2 \mu \ell^3 \hat{\omega}^2 (\hat{\alpha}+\omega_o) h_1 t}{I\hat{k}_p}\right]^{\frac{1}{2}}} . \qquad (4.28)
$$

The further approximations can be calculated in precisely the same manner. It may be noted that the first order solution (4.28) of this method is, as mentioned in chapter III, also obtained by the energy sink method.

A criterion for the validity of this method can be set up by using Eqs. (4.27) : This method is applicable if the right hand sides of Eqs. (4.27) are small. Inaccuracy may arise if the so called resonance conditions are satisfied :

$$\frac{\delta}{k_p}\ , \quad \frac{(k_p-2\hat{\alpha})}{k_p} \lesssim 0\ [\ (\frac{\epsilon\hat{\omega}}{k_p})^2\ ]\ . \tag{4.29}$$

When the conditions (4.29) are satisfied, a large nonlinear energy transfer takes place between the vibrations of the appendages and nutational body motions, which results in a beating between the two modes. As a result, the basic assumption of this method, that the amplitude of the nutational body motion is a slowly varying function of time, no longer holds.

Let us now consider in more detail the case where the resonance conditions are satisfied. We start afresh from Eqs. (4.17) and seek a solution in the the form

$$\left.\begin{aligned} \omega_1 &= \hat{\omega}(ae^{ik_p t/2} + a^* e^{-ik_p t/2}) \\ \omega_2 &= -i\hat{\omega}(ae^{ik_p t/2} - a^* e^{-ik_p t/2}) \\ P &= ce^{ik_p t} + c^* e^{-ik_p t} \end{aligned}\right\} \tag{4.30}$$

with the additional condition

$$\dot{c}e^{ik_p t} + \dot{c}^* e^{-ik_p t} = 0\ . \tag{4.31}$$

Equations of motion (4.17), on substitution of Eqs. (3.30), (3.31), are reduced to

$$\dot{a} = -i\Delta\ (a - a^* e^{-ik_p t}) + (\frac{\epsilon\mu\ell^3}{I})\ [\ -(\frac{k_p}{2} + \omega_o)\, a^* c$$

$$+ (\frac{3}{2}k_p - \omega_o)\, a^* c^* e^{-2ik_p t} + i\dot{a}^*(c + c^* e^{-2ik_p t})\ ]$$

$$\dot{c} = -\delta\,(c - c^{*} e^{-ik_p t}) + \frac{2\epsilon\hat{\omega}^2}{k_p}(a^2 - a^{*2} e^{-2ik_p t}) \tag{4.32}$$

where

$$\Delta = \frac{k_p}{2} - \hat{\alpha}\,.$$

The variables a and c are slow varying functions since $\dot{a}$, $\dot{c} \sim O(\epsilon)$, so that the method of averating [35] can be successfully applied to achieve an approximate solution of Eqs. (4.32). Let us expand the variables a, c in the power series of ϵ as follows :

$$\left.\begin{aligned}
a &= \hat{a} + \sum_{n=1}^{\infty} \epsilon^n F_{\hat{a}}^{(n)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*, t)\\
c &= \hat{c} + \sum_{n=1}^{\infty} \epsilon^n F_{\hat{c}}^{(n)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*, t)\\
\dot{\hat{a}} &= \sum_{n=1}^{\infty} G_{\hat{a}}^{(n)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*)\\
\dot{\hat{c}} &= \sum_{n=1}^{\infty} G_{\hat{c}}^{(n)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*)\,.
\end{aligned}\right\} \tag{4.33}$$

Substituting Eqs. (4.33) into Eqs. (4.32), taking into account the relations

$$\frac{\delta}{k_p},\ \frac{(k_p - 2\hat{\alpha})}{k_p} \lesseqqgtr 0\ \left[\left(\frac{\epsilon\hat{\omega}}{k_p}\right)^2\right]$$

and equating the terms of the same order, we get

$$\begin{aligned}
\epsilon G_{\hat{a}}^{(1)} + \epsilon \frac{\partial F_{\hat{a}}^{(1)}}{\partial t} &= -\left(\frac{\mu\ell^3\epsilon}{I}\right)\left[\left(\frac{k_p}{2} + \omega_o\right)\hat{a}^*\hat{c}\right.\\
&\qquad \left. - \left(\frac{3}{2}k_p - \omega_o\right)\hat{a}^*\hat{c}^* e^{-2ik_p t}\right]
\end{aligned} \tag{4.34.a}$$

$$\epsilon G_{\hat{c}}^{(1)} + \epsilon \frac{\partial F_{\hat{c}}^{(1)}}{\partial t} = \left(\frac{2\epsilon\hat{\omega}^2}{k_p}\right)\left[\hat{a}^2 - \hat{a}^{*2} e^{-2ik_p t}\right]$$

$$
\left.
\begin{aligned}
\epsilon^2 G_{\hat a}^{(2)} + \epsilon^2 \frac{\partial F_{\hat a}^{(2)}}{\partial t} = & -\epsilon^2 \Big(G_{\hat a}^{(1)} \frac{\partial}{\partial \hat a} + G_{\hat a}^{(1)*} \frac{\partial}{\partial \hat a^*} + G_{\hat c}^{(1)} \frac{\partial}{\partial \hat c} \\
& + G_{\hat c}^{(1)*} \frac{\partial}{\partial \hat c^*} \Big) F_{\hat a}^{(1)} - i\Delta \hat a + i\Delta \hat a^* e^{-ik_p t} \\
& - \Big(\frac{\mu \ell^3 \epsilon^2}{I}\Big)\Big(\frac{k_p}{2} + \omega_o\Big)\big(\hat a^* F_{\hat c}^{(1)} + \hat c F_{\hat a}^{(1)*}\big) \\
& + \Big(\frac{\mu \ell^3 \epsilon^2}{I}\Big)\Big(\frac{3k_p}{2} - \omega_o\Big)\big(\hat a^* F_{\hat c}^{(1)*} + \hat c^* F_{\hat a}^{(1)*}\big) e^{-2ik_p t} \\
& + i\Big(\frac{\mu \ell^3 \epsilon^2}{I}\Big)\Big(G_{\hat a}^{(1)*} + \frac{\partial F_{\hat a}^{(1)*}}{\partial t}\Big)\big(\hat c + \hat c^* e^{-2ik_p t}\big) \\
\epsilon^2 G_{\hat c}^{(2)} + \epsilon^2 \frac{\partial F_{\hat c}^{(2)}}{\partial t} = & -\epsilon^2 \Big(G_{\hat a}^{(1)} \frac{\partial}{\partial \hat a} + G_{\hat a}^{(1)*} \frac{\partial}{\partial \hat a^*} + G_{\hat c}^{(1)} \frac{\partial}{\partial \hat c} \\
& + G_{\hat c}^{(1)*} \frac{\partial}{\partial \hat c^*} \Big) F_{\hat c}^{(1)} - \delta\big(\hat c - \hat c^* e^{-2ik_p t}\big) \\
& + \Big(\frac{4\epsilon^2 \hat\omega^2}{k_p}\Big)\big(\hat a F_{\hat a}^{(1)} - \hat a^* F_{\hat a}^{(1)*} e^{-2ik_p t}\big).
\end{aligned}
\right\}
\qquad (4.34.b)
$$

. . .

Application of the method of averaging to Eqs. (4.34.a), (4.34.b) leads to the following second order equations for a and c

$$
\left.
\begin{aligned}
\dot{\hat a} &= -\epsilon \alpha_o \hat a^* \hat c - i\Delta \hat a + i\epsilon^2 \alpha_1 \hat a \hat c \hat c^* + i\epsilon^2 \alpha_2 \hat a^2 \hat a^* \\
\dot{\hat c} &= \epsilon \beta_o \hat a^2 - \delta \hat c + i\epsilon^2 \beta_1 \hat a \hat a^* \hat c
\end{aligned}
\right\}
\qquad (4.35)
$$

where

$$
\alpha_o = \Big(\frac{\mu \ell^3}{I}\Big)\Big(\frac{k_p}{2} + \omega_o\Big)
$$

$$
\alpha_1 = -\Big(\frac{\mu \ell^3}{I}\Big)^2 \Big(\frac{k_p}{2} + \omega_o\Big)^2 \frac{1}{2k_p}
$$

$$\alpha_2 = \left(\frac{\mu \ell^3}{I}\right)\left(\frac{\hat{\omega}}{k_p}\right)^2\left(\frac{3}{2}k_p - \omega_o\right)$$

$$\beta_o = 2\frac{\hat{\omega}^2}{k_p}$$

$$\beta_1 = 2\left(\frac{\mu \ell^3}{I}\right)\left(\frac{\hat{\omega}}{k_p}\right)^2\left(\frac{3}{2}k_p - \omega_o\right).$$

We seek a solution in the form

$$\left.\begin{aligned} \hat{a} &= re^{i(\text{Ⓗ} - \Delta t)} \\ \hat{c} &= qe^{i\Gamma} \end{aligned}\right\} \qquad (4.36)$$

where r is proportional to the amplitude of the angular velocity components ω_1, ω_2 (and is also proportional to the amplitude of the nutational body motion), q is proportional to the amplitude of the vibrations of the appendages. Equations (4.35), then, take the forms

$$\dot{r} = -\epsilon\alpha_o rq\cos\lambda \qquad (4.37.a)$$

$$\dot{q} = \epsilon\beta_o r^2\cos\lambda - \delta q \qquad (4.37.b)$$

$$\dot{\text{Ⓗ}} = \epsilon\,\alpha_o q\sin\lambda + \epsilon^2(\alpha_1 q^2 + \alpha_2 r^2) \qquad (4.37.c)$$

$$\dot{\Gamma} = \frac{\epsilon\beta_o r^2}{q}\sin\lambda + \epsilon^2\beta_1 r^2 \qquad (4.37.d)$$

where

$$\lambda = 2\,\text{Ⓗ} - 2\Delta t - \Gamma$$

and the initial conditions (4.14) become

$$\left.\begin{array}{lll} t = 0\,, & r = \dfrac{1}{2}\,, & q = 0 \\ & Ⓗ = 0\,, & \Gamma = 0\,. \end{array}\right\} \tag{4.38}$$

In what follows, we shall discuss the damping characteristics of nutational body motions due to a nonlinear internal resonance on the basis of the averaged equations (4.37) and the initial conditions (4.38). First, let us assume $\delta = 0$. Then, Eqs. (4.37) become

$$\dot{r} = -\epsilon\,\alpha_o rq \cos\lambda \tag{4.39.a}$$

$$\dot{q} = \epsilon\beta_o r^2 \cos\lambda \tag{4.39.b}$$

$$\dot{Ⓗ} = \epsilon\,\alpha_o q \sin\lambda + \epsilon^2(\alpha_1 q^2 + \alpha_2 r^2) \tag{4.39.c}$$

$$\dot{\Gamma} = \frac{\epsilon\beta_o r^2}{q} \sin\lambda + \epsilon^2 \beta_1 r^2\,. \tag{4.39.d}$$

Using Eqs. (4.39.a), (4.39.b), we obtain, upon integration,

$$\epsilon\beta_o r^2 + \epsilon\alpha_o q^2 = \epsilon\beta_o r_o^2 \tag{4.40}$$

where the constant of integration r_o^2 is proportional to the total energy of the system. On the other hand, using Eqs. (4.39.a), (4.39.c), (4.39.d) we derive the following differential relation

$$\begin{aligned} &[\,\epsilon\,(2\alpha_o q - \frac{\beta_o r^2}{q}) \sin\lambda + \epsilon^2(2\alpha_1 q^2 + 2\alpha_2 r^2 - \beta_1 r^2) \\ &\qquad -2\Delta\,]\;dr + \epsilon\alpha_o rq \cos\lambda\; d\lambda = 0\,. \end{aligned} \tag{4.41}$$

Integrating Eq. (4.41) with conditions (4.38), we obtain

$$\epsilon\,(\beta_o r_o^2 - \alpha_o q^2)\, q \sin\lambda - \frac{1}{2}\epsilon^2 \alpha_1 q^4 + \Delta q^2 = 0. \tag{4.42}$$

Then, using Eqs. (4.39.b), (4.40), (4.42) we obtain the differential relation

$$dt = \frac{dQ}{\pm 2\left\{Q\,(\epsilon^2 \beta_o r_o{}^2 - \alpha_o Q)^2 - [\,-\Delta Q + \frac{\alpha_1 Q^2}{2}\,]^2\right\}^{\frac{1}{2}}} \tag{4.43}$$

where $Q = (\epsilon q)^2$

Zeros of the quartic expression under the radical in the integrand of Eq. (4.43) are given approximately by

$$\left.\begin{aligned} Q_o &= 0 \\ Q_1 &= A^2\,(\,1 - \frac{W}{A}) \\ Q_2 &= A^2\,(\,1 + \frac{W}{A}) \\ Q_3 &= (\frac{4k_p}{\alpha_o})^2 \end{aligned}\right\} \tag{4.44}$$

where

$$A^2 = \epsilon^2 r_o{}^2 (\frac{\beta_o}{\alpha_o}) \qquad (A > 0)$$

$$W^2 = (\,\frac{\epsilon^2 \beta_o r_o{}^2}{4k_p} + \frac{\Delta}{\alpha_o})^2 \qquad (W > 0)\,.$$

Since the values of $\frac{Q_1}{Q_2}, \frac{Q_2}{Q_3}$ are of the order $O\,[\,(\frac{\epsilon\hat{\omega}}{k_p})^2\,]$ and the physically significant solution corresponds to the range $Q_o \leq Q \leq Q_1$, the quartic expression can be approximated by the following cubic expression :

$$\alpha_o{}^2 Q\,(\,Q - Q_1\,)\,(\,Q - Q_2\,)\,.$$

Then, Eq. (4.43) reduces to

$$dt = \frac{dQ}{\pm\, 2\,[\,\alpha_o{}^2 Q\,(\,Q - Q_1\,)\,(\,Q - Q_2\,)\,]^{\frac{1}{2}}} \tag{4.45}$$

Equation (4.45) can be integrated directly by means of elliptic integrals and a solution is given in the form

$$\left.\begin{aligned} r^2 &= r_o^2 \left[1 - \kappa\, \mathrm{sn}^2 \left(\frac{\alpha_o A t}{\kappa^{\frac{1}{2}}}, \kappa \right) \right] \\ q^2 &= r_o^2 \left(\frac{\beta_o}{\alpha_o} \right) \kappa\, \mathrm{sn}^2 \left(\frac{\alpha_o A t}{\kappa^{\frac{1}{2}}}, \kappa \right) \end{aligned}\right\} \tag{4.46}$$

where

$$\kappa = 1 - \frac{W}{A} .$$

The least period of the functions r and q is given by

$$J = \frac{2\kappa^{\frac{1}{2}} K(\kappa)}{\alpha_o A} \tag{4.47}$$

where K (κ) is the complete elliptic integral of the first kind

$$K(\kappa) = \int_o^{\frac{\pi}{2}} \frac{d\rho}{(1 - \kappa^2 \sin^2 \rho)^{\frac{1}{2}}}$$

We next proceed to an approximate solution of Eqs. (4.37) valid for finite values of the parameter δ . By making δ finite, we can expect that the total energy of the system decreases slowly as time increases : The quantity r_o becomes slowly varying function of time. Combination of Eqs. (4.37.a) and (4.37.b) leads to

$$\frac{d}{dt} \left[r^2 + \left(\frac{\alpha_o}{\beta_o} \right) q^2 \right] = -2\delta \left(\frac{\alpha_o}{\beta_o} \right) q^2 . \tag{4.48}$$

Substituting Eqs. (4.46) into Eq. (4.48) and taking into account the fact that r_o is a slowly varying function of time, we obtain the following equation :

$$\frac{dr_o}{dt} = -\delta r_o \kappa\, \mathrm{sn}^2 \left(\frac{\alpha_o A t}{\kappa^{\frac{1}{2}}}, \kappa \right) . \tag{4.49}$$

Since the parameter δ is small, applying the method of averaging once more to Eq. (4.49), we can obtain an approximate solution of r_o. Application of the method of averaging to Eq. (4.49) gives the first order equation for r_o :

$$\left.\begin{aligned} r_o &= \hat{r}_o \\ \dot{\hat{r}}_o &= -\left(\frac{\delta \hat{r}_o}{\kappa}\right)\left(1 - \frac{E(\kappa)}{K(\kappa)}\right) \end{aligned}\right\} \tag{4.50}$$

where $E(\kappa)$ is the complete elliptic integral of the second kind

$$E(\kappa) = \int_0^{\frac{\pi}{2}} (1 - \kappa^2 \sin^2 \rho)^{\frac{1}{2}}\, d\rho \, .$$

Neglecting small terms as compared with those of the order $O\left[\left(\frac{\epsilon \hat{\omega}}{k_p}\right)^2\right]$, we obtain a solution in the form

$$\hat{r}_o = \frac{1}{2} e^{-\delta^* t} \tag{4.51}$$

where

$$\delta^* = \frac{\delta}{\hat{\kappa}}\left(1 - \frac{E(\hat{\kappa})}{K(\hat{\kappa})}\right)$$

$$\hat{\kappa} = 1 - \frac{\hat{W}}{\hat{A}}$$

$$\hat{A}^2 = \epsilon^2 \left(\frac{l}{2}\right)^2 \left(\frac{\beta_o}{\alpha_o}\right), \qquad \hat{W}^2 = \left(\frac{\epsilon^2 \left(\frac{l}{2}\right)^2 \beta_o}{4k_p} + \frac{\Delta}{\alpha_o}\right)^2 .$$

It must be remarked that an amplitude of nutational body motions decays exponentially on the average due to the nonlinear internal resonance between the vibrations of the appendages and nutational body motions. This solution is valid so long as the quantity r_o decreases gradually over time interval of the period

J, i.e.,

$$\frac{J}{(1/\delta^*)} \ll 1 .$$

This condition means that

$$\frac{J}{(1/\delta^*)} \sim \frac{\delta K(\hat{\kappa})}{\alpha_0 \hat{A} \hat{\kappa}^{\frac{1}{2}}} \sim \epsilon\hat{\omega}K(\hat{\kappa}) \ll 1 .$$

In a resonance case, this condition is always fulfilled since

$$\lim_{\kappa \to 1-0} K(\kappa) = O\,[\ell n(1 - \kappa)].$$

Typical curves of the amplitude of the angular velocity component ω_1 are shown in Figs. (4.2), (4.3). In the same figures, numerical solutions of Eqs. (4.17) are also shown to check the results. In Fig. 3.2 is shown the amplitude of ω_1 with the time t as a parameter by means of the formula (4.28) for a typical nonresonance case where the system parameters are given as follows :

$$\hat{\alpha} = 1.0 \qquad \frac{\mu \ell^3}{I} = 0.67 \qquad \frac{k_p - 2\hat{\alpha}}{k_p} = -0.0025$$

$$\frac{\epsilon\hat{\omega}}{k_p} = 0.057 \qquad \frac{\delta}{k_p} = 0.050 .$$

Figure 3.2 shows that in a nonresonance case Eq. (4.28) is a good approximate solution for the amplitude of ω_1. In Fig. 3.3.a is shown the amplitude of ω_1 for a resonance case where system parameters are given as follows :

$$\hat{\alpha} = 1.0 \qquad \frac{\mu \ell^3}{I} = 0.67 \qquad \frac{k_p - 2\hat{\alpha}}{k_p} = -0.0025$$

$$\frac{\epsilon\omega}{k_p} = -0.057 \qquad \frac{\delta}{k_p} = 0.0017 .$$

An envelope of a long period motion of the amplitude of ω_1 obtained by means of the formula (4.51) is in good agreement with a numerically computed solution. Figure 4.3.b shows an amplitude of ω_1 for large Δ, i.e., $\frac{k_p - 2\alpha}{k_p} = 0.332$ (nonresonance case) ; all other parameters are the same as in Fig. 4.3.a. It can be concluded from Figs. 4.3 that the amplitude of ω_1 decays very rapidly due to the resonance between the vibrations of the appendages and nutational body motions.

4.4 Conclusions

The heavy damping characteristics of nutational body motions due to the nonlinear internal resonance between the vibrations of the appendages and nutational body motions are investigated analytically on the basis of the method of averaging. The analysis shows a marked increase of resonant over nonresonant cases in nutation damping : The amplitude of nutational body motions decays very rapidly due to a resonance between the vibrations of the appendages and nutational body motions. Moreover, the nutational body motion is shown to decay exponentially on the average due to the nonlinear internal resonance.

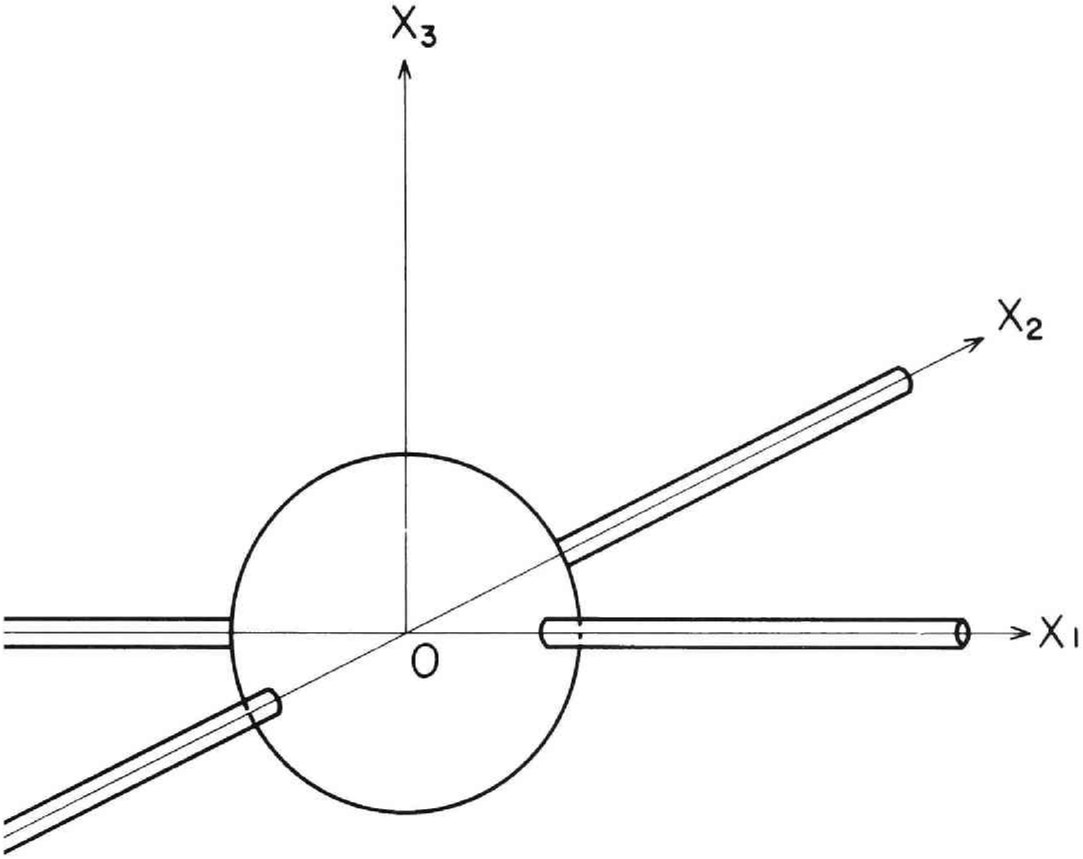

Fig. 4.1 Spacecraft configuration

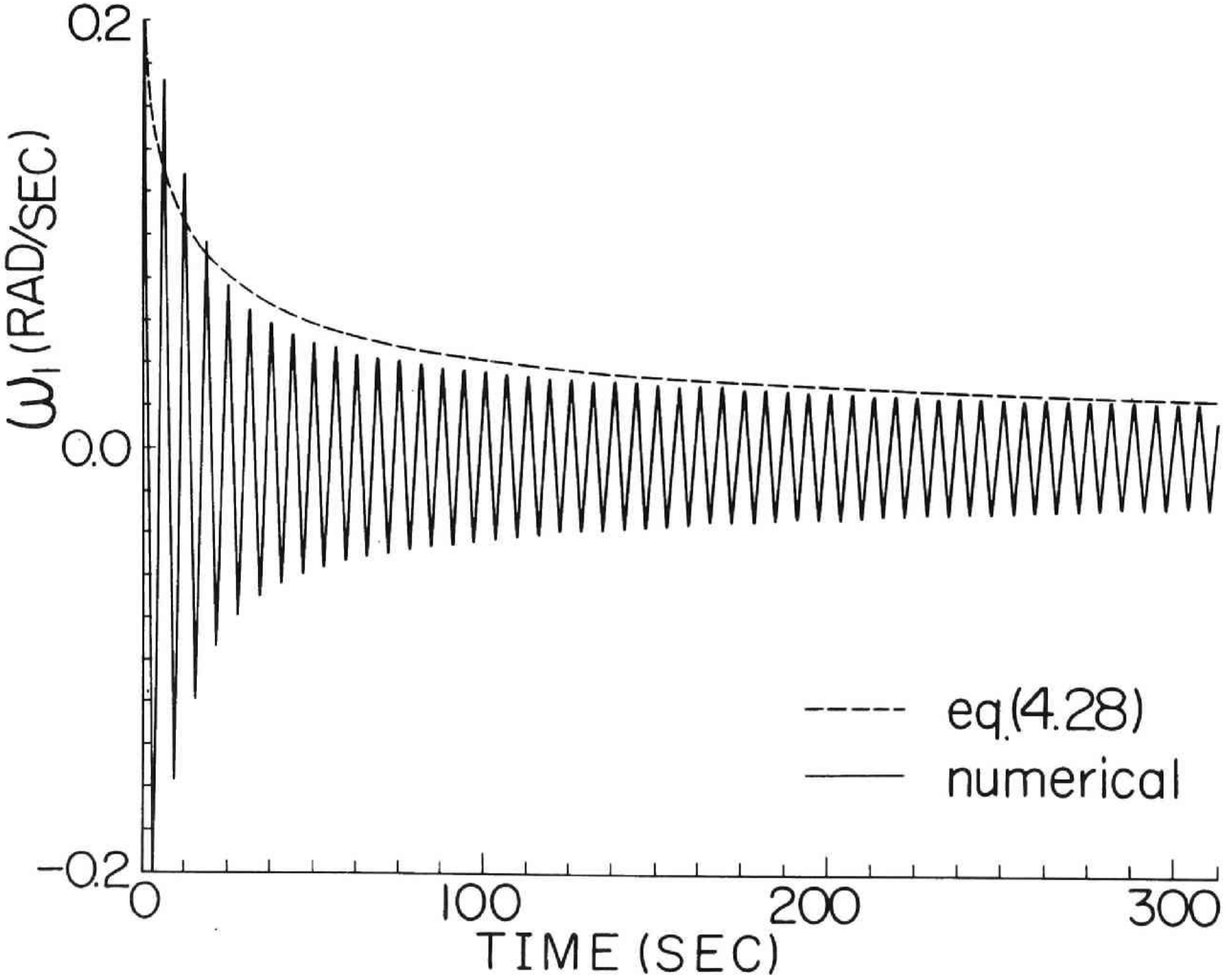

Fig. 4.2 Plot of ω_1 against time (nonresonance case)

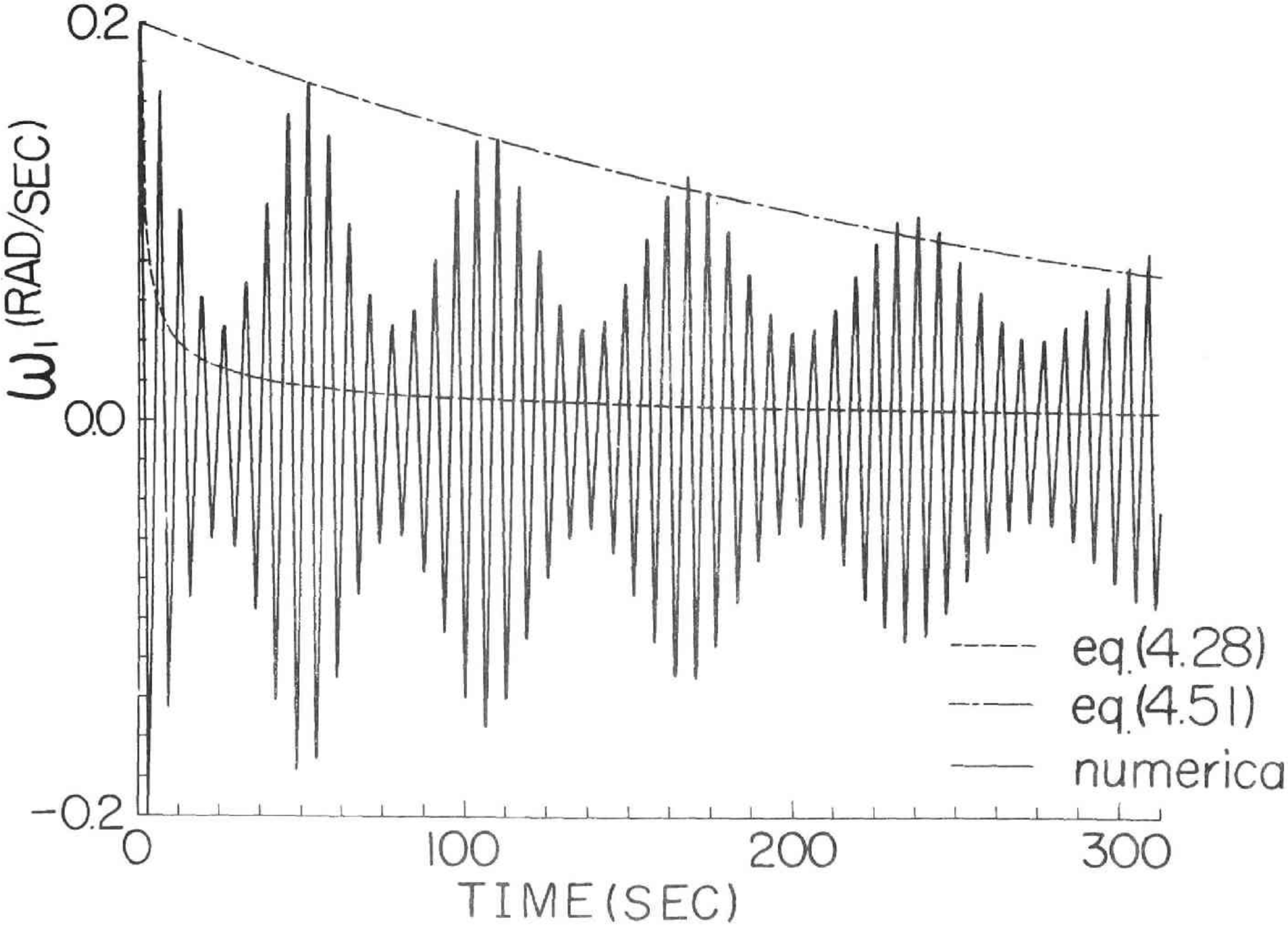

Fig. 4.3.a Plot of ω_1 against time (resonance case)

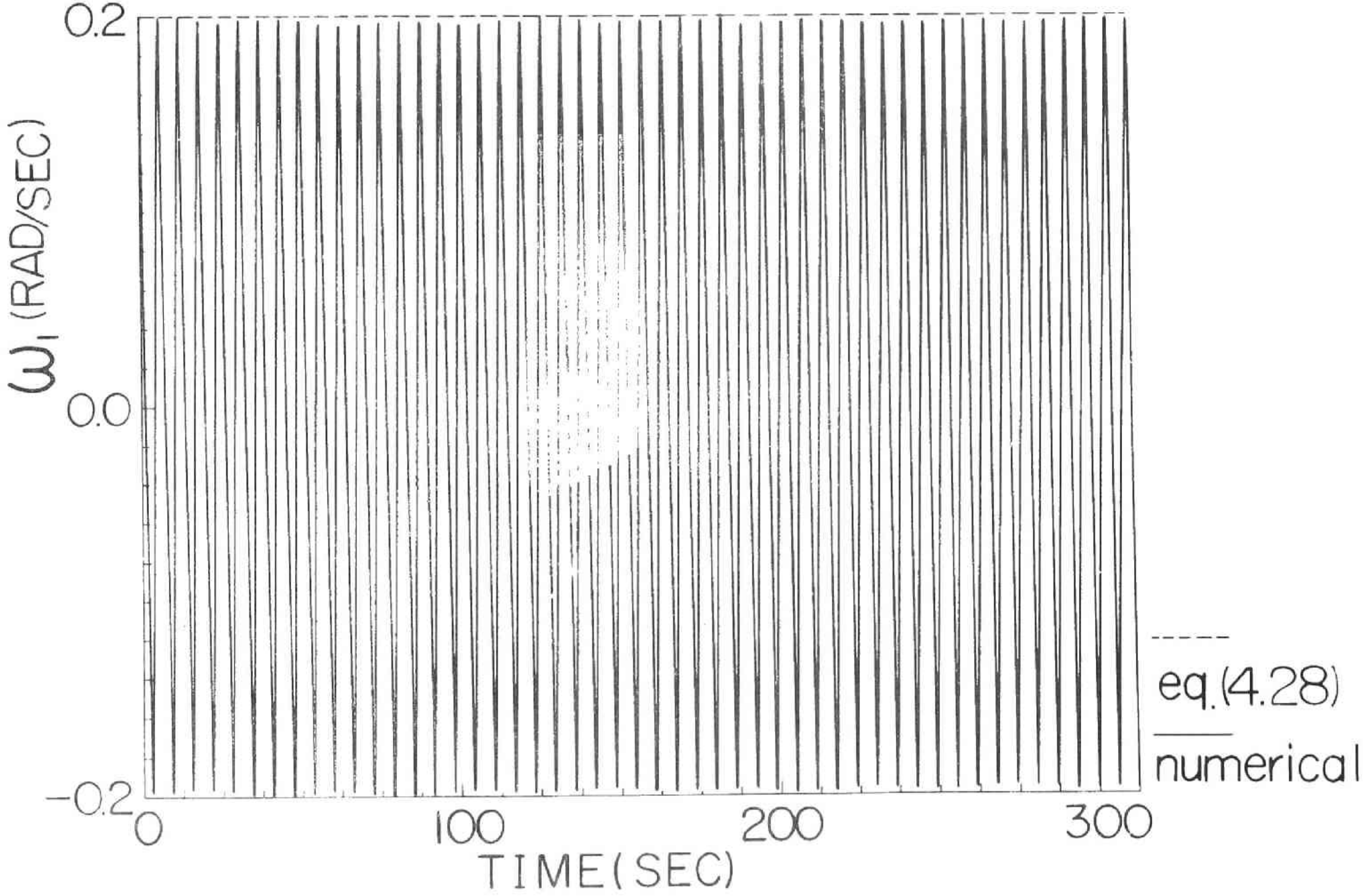

Fig. 4.3.b Plot of ω_1 against time (nonresonance)

CHAPTER V

THERMALLY INDUCED NUTATIONAL BODY MOTION OF A SPINNING SPACECRAFT HAVING FLEXIBLE APPENDAGES

5.1 Introduction

Current trends in the development of spacecraft show an increasing reliance on large, highly flexible appendages as antennas or solar arrays. For this class of spacecraft, it is possible that the flexible appendages can interact unfavorably with the environment and this interaction leads to large attitude errors. In fact, recent flight data have shown much anomalous behavior due to flexibility of appendages.(17)

The objective of this chapter is to predict that a class of spinning spacecraft with flexible appendages may exhibit an anomalous behavior, a steady nutational body motion which is caused by an interaction of flexible appendages with solar radiation. The problem to be analyzed is as follows : A spinning spacecraft which has flexible appendages is considered (Fig. 5.1). The appendages are assumed to lie in a plane normal to the spin axis. Solar radiation is assumed normal to the spin axis. When this class of spacecraft is exposed to solar radiation, vibrations of the appendages in a spin plane are induced at a spin frequency by solar heating. These vibrations cause a periodic variation of the moments of inertia of the spacecraft. Usually, the influence of this variation of the moments of inertia upon nutational body motions is small. However, if the spin velocity is approximately equal to twice the frequency of the nutational body motion (this means that the ratio of the spin moment of inertia to the transverse moment of inertia is nearly equal to 3/2 for the vehicle in its undeformed shape) the amplitude of the nutational body motion builds up to larger values through the instrumentality of nonlinear parametric excitation. The method of averaging is applied to this problem to obtain the amplitude of the nutational body motion analytically, and the stability of the motion is examined in detail.

5.2 Equations of Motion

Consider a symmetrical spinning spacecraft composed of a heavy central rigid body and light-weight flexible appendages (Fig. 5.1). The reference axes (X_1, X_2, X_3) are assumed to be parallel to the principal axes of the undeformed total configuration (X_3 axis coincides with the spin axis) and have the origin at the mass center of the undeformed total configuration. For an appendage i, an axis system (ζ_i, η_i, ξ_i) is defined so that the appendage is coincident with the ζ_i axis at the undeflected condition, and the ξ_i axis coincides with the spin axis. Let the angle of rotation from (X_1, X_2, X_3) to (ζ_i, η_i, ξ_i) be γ_i and let the angular velocity of the (X_1, X_2, X_3) axes have the components (ω_1, ω_2, ω_3) in the (X_1, X_2, X_3) reference frame.

In the present study, the mass center is assumed to remain fixed at the origin of the axis system (X_1, X_2, X_3). As mentioned above, thermally induced vibrations of the appendages in the spin plane cause a nutational body motion of the spacecraft. The nutational body motion, on the other hand, induces out of plane vibrations of the appendages. The out of plane vibrations can interact with the nutational body motion, but, in what follows, we shall neglect the out of plane vibrations for the sake of simplicity. This simplification does not alter the essential features of the phenomena discussed herein. Furthermore, we shall, for the moment, confine ourselves to the case where the effect of solar heating is so small that the induced vibrations of the appendages are small.

Then, the total kinetic energy T takes the form,

$$2T = I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 + \sum_{i=1}^{N} \mu_i \int_0^{\ell_i} \Big\{ \dot{u}_i^2 + 2\omega_3 \zeta_i \dot{u}_i$$

$$+ S2\gamma_i\, \omega_1^2 \zeta_i u_i - S2\gamma_i \omega_2^2 \zeta_i u_i + \omega_3^2 \,[\, u_i^2$$

$$- \tfrac{1}{2} (\ell_i^2 - \zeta_i^2) \left(\frac{\partial u_i}{\partial \zeta_i}\right)^2] - 2C2\gamma_i \omega_1 \omega_2 \zeta_i u_i \Big\} d\zeta_i \tag{5.1}$$

where I and I_3 are the moments of inertia of the undeformed total configuration about X_1 (or X_2), X_3 axes, respectively, u_i a deflection of an appendage i, ℓ_i the length of the appendage i and $S2\gamma_i = \sin 2\gamma_i$, $C2\gamma_i = \cos 2\gamma_i$.

The elastic potential energy U which arises from the strain energy due to the appendage deformations is given by

$$2U = \sum_{i=1}^{N} B_i \int_0^{\ell_i} \left(\frac{\partial^2 u_i}{\partial \zeta_i^2}\right)^2 d\zeta_i \tag{5.2}$$

where B_i is the bending stiffness of an appendage i.

The energy dissipation which results from elastic deformations of the appendages is represented by Rayleigh's dissipation function F, which is given by

$$F = \sum_{i=1}^{N} \mu_i \delta_i \int_0^{\ell_i} \dot{u}_i^2 \, d\zeta_i \tag{5.3}$$

where δ_i is the damping ratio for an appendage i.

We shall represent the elastic deformations u_i by the following series :

$$u_i(\zeta_i, t) = \ell_i \sum_{n=1}^{\infty} P_{in}(t)\, E_n(\hat{\zeta}) \tag{5.4}$$

where $E_n(\hat{\zeta})$ are normal modes associated with a cantilever and P_{in} are the corresponding generalized coordinates. The normal modes $E_n(\hat{\zeta})$ are defined by

$$\left.\begin{aligned} &\frac{d^4 E_n(\hat{\zeta})}{d\hat{\zeta}^4} - \lambda_n{}^4 E_n(\hat{\zeta}) = 0 \\ &\hat{\zeta} = 0, \qquad E_n(\hat{\zeta}) = \frac{dE_n(\hat{\zeta})}{d\hat{\zeta}} = 0 \\ &\hat{\zeta} = 1, \qquad \frac{d^2 E_n(\hat{\zeta})}{d\hat{\zeta}^2} = \frac{d^3 E_n(\hat{\zeta})}{d\hat{\zeta}^3} = 0 \end{aligned}\right\} \tag{5.5}$$

where λ_n^4 are the eigenvalues of the normal modes $E_n(\hat{\zeta})$ and $\hat{\zeta} = \frac{\zeta_i}{\ell_i}$.

In addition, they are normalized such that

$$\int_0^1 E_n(\hat{\zeta})\, E_m(\hat{\zeta}) d\hat{\zeta} = \delta_{n,m} \tag{5.6}$$

where $\delta_{n,m}$ is Kronecker's delta. In this study, attention will be paid to the case in which the appendages are excited near the first resonance frequency. Then, the deflections, u_i, can be approximated by the first mode, i.e.,

$$u_i(\zeta_i, t) = \ell_i\, P_{i1}(t) E_1(\hat{\zeta}) \tag{5.4'}$$

Substituting Eq. (5.4)' into Eqs. (5.1), (5.2), (5.3) and neglecting the suffix 1, we obtain

$$\begin{aligned} 2T = I(\omega_1{}^2 + \omega_2{}^2) + I_3\omega_3{}^2 + \sum_{i=1}^{N} \mu_i \ell_i{}^3\, [\, \dot{P}_i{}^2 + 2\epsilon\,\omega_3 \dot{P}_i \\ + S2\gamma_i \epsilon\omega_1^2\, P_i - S2\,\gamma_i\, \epsilon\omega_2{}^2 P_i + (1-\beta)\,\omega_3{}^2 P_i{}^2 \end{aligned}$$

$$-2C2\gamma_i \epsilon \omega_1 \omega_2 P_i \;]$$

(5.7)

$$2U = \sum_{i=1}^{N} \frac{\lambda^4 B_i}{\ell_i} P_i^2 \tag{5.8}$$

$$F = \sum_{i=1}^{N} \mu_i \ell_i \delta_i \dot{P}_i^2 \tag{5.9}$$

where

$$\epsilon = \int_0^1 \hat{\xi} E(\hat{\xi}) \, d\hat{\xi} = -0.5688$$

$$\beta = \tfrac{1}{2} \int_0^1 (1 - \hat{\xi}^2)\left(\frac{dE(\hat{\xi})}{d\hat{\xi}}\right)^2 d\hat{\xi} = 1.193 \, .$$

Since the coordinates ω_1, ω_2, ω_3 are so-called quási-coordinates, the corresponding equations of motion are [32]

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + \omega_2 \left(\frac{\partial T}{\partial \omega_3}\right) - \omega_3 \left(\frac{\partial T}{\partial \omega_2}\right) = N_1 \\
&\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) + \omega_3 \left(\frac{\partial T}{\partial \omega_1}\right) - \omega_1 \left(\frac{\partial T}{\partial \omega_3}\right) = N_2 \\
&\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) + \omega_1 \left(\frac{\partial T}{\partial \omega_2}\right) - \omega_2 \left(\frac{\partial T}{\partial \omega_1}\right) = N_3
\end{aligned}\right\} \tag{5.10}$$

where N_1, N_2, N_3 are the external torque components about the X_1, X_2, X_3 axes, respectively. The Lagrange equations of motion for the generalized coordinates P_i take the form [32]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{P}_i}\right) + \left(\frac{\partial F}{\partial \dot{P}_i}\right) - \left(\frac{\partial L}{\partial P_i}\right) = Q_i \tag{5.11}$$

where the Lagrangian L is given by L=T - U, Q_i are the generalized forces arising from external sources. The attitude motion of the spacecraft is affected by external torques and forces of various forms. The present chapter considers nutational body motions induced by solar heating, so that other external torques and forces are neglected.

Solar heating produces a temperature difference across an appendage. The temperature distribution induces a thermal strain along the appendage length. Etkin and Hughes [18] state that, on the assumption that solar radiation is normal to the spin axis, the steady periodic thermal bending moment with the spin frequency at any section of an appendage i approximated by

$$\left.\begin{aligned} M_{i_1} &= 0 \\ M_{i_2} &= 0 \\ M_{i_3} &= f_{oi}\cos(\tau+\tau_i) \end{aligned}\right\} \qquad (5.12)$$

where M_{i1}, M_{i2}, M_{i3} are the components of the thermal bending moment about ζ_i, η_i, ξ_i axes, respectively, f_{oi} is a constant, $\tau_i = \tau_o + \gamma_i$ (τ_o is a constant phase lag), and $\frac{d\tau}{dt} = \omega_3$.

The work done by the thermal bending moment in an arbitrary displacement δu_i takes the form

$$\delta W = \sum_{i=1}^{N} f_{oi}\cos(\tau + \tau_i) \int_{o}^{\ell_i} \left(\frac{\partial^2(\delta u_i)}{\partial \zeta_i^2}\right) d\zeta_i. \qquad (5.13)$$

Upon substitution of Eq. (5.4)' into Eq. (5.13) and neglecting the suffix 1, we find

$$\delta W = \sum_{i=1}^{N} f_{oi}\cos(\tau+\tau_i)\frac{E'(1)}{\ell_i}\,\delta P_i. \qquad (5.14)$$

Then, the generalized forces Q_i arising from the effect of solar heating are obtained from Eqs. (5.14) as

$$Q_i = f_{oi} \cos(\tau + \tau_i) \frac{E'(1)}{\ell_i} . \tag{5.15}$$

On the other hand, the action of the solar radiation pressure produces torques on the spacecraft. The torques, however, are so small that we neglect the torques in the following analysis.

Hence, it holds that

$$\left.\begin{aligned} N_1 &= 0 \\ N_2 &= 0 \\ N_3 &= 0 \end{aligned}\right\} \tag{5.16}$$

Substituting Eqs. (5.7), (5.8), (5.9), (5.15), (5.16) into Eqs. (5.10), (5.11), we obtain a system of equations of motion as follows :

$$\left.\begin{aligned} \dot{\omega}_1 + \alpha\omega_2 &= -\sum_{i=1}^{N} \frac{\epsilon\mu_i\ell_i^3}{I} [\, \omega_2 \dot{P}_i + S2\gamma_i (P_i \dot{\omega}_1 + \dot{P}_i \omega_1) \\ &\qquad - C2\gamma_i (\dot{P}_i \omega_2 + P_i \dot{\omega}_2) + C2\gamma_i \omega_1 \omega_3 P_i + S2\gamma_i \omega_2 \omega_3 P_i \,] \\ \dot{\omega}_2 - \alpha\omega_1 &= \sum_{i=1}^{N} \frac{\epsilon\mu_i\ell_i^3}{I} [\, \omega_1 \dot{P}_i + C2\gamma_i (\dot{P}_i \omega_1 + P_i \dot{\omega}_i) \\ &\qquad + S2\gamma_i (\dot{P}_i \omega_2 + P_i \dot{\omega}_2) - S2\gamma_i \omega_1 \omega_3 P_i \\ &\qquad + C2\gamma_i \omega_2 \omega_3 P_i \,] \\ \dot{\omega}_3 &= -\sum_{i=1}^{N} \frac{\epsilon\mu_i\ell_i^3}{I_3} \ddot{P}_i \end{aligned}\right\} \tag{5.17}$$

$$\ddot{P}_i + 2\delta_i \dot{P}_i + k_i^2 P_i = -\epsilon\dot{\omega}_3 + \frac{\epsilon}{2}(S2\gamma_i\omega_1^2 - 2C2\gamma_i\omega_1\omega_2 - S2\gamma_i\omega_2^2) + \frac{f_{oi}E'(1)}{\mu_i \ell_i^4} \cos(\tau + \tau_i)$$

$$\dot{\tau} = \omega_3$$

where

$$\alpha = \left(\frac{I_3}{I} - 1\right)\omega_3, \quad k_i^2 = \frac{\lambda^4 B_i}{\mu_i \ell_i^4} + (\beta - 1)\omega_3^2 .$$

We introduce a new complex variable a as follows :

$$\left.\begin{aligned} \omega_1 &= (ae^{i\tau/2} + a^* e^{-i\tau/2}) \\ \omega_2 &= -i(ae^{i\tau/2} - a^* e^{-i\tau/2}) \end{aligned}\right\} \qquad (5.18)$$

where a* is a complex conjugate of a. Substituting Eqs. (5.18) into Eqs. (5.17), we have

$$\dot{a} = i\left(\alpha - \frac{\omega_3}{2}\right) a + \epsilon \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{2I} (2ia\dot{P}_i + 2ia^*\dot{P}_i e^{i2\gamma_i} e^{-i\tau} - \omega_3 a^* P_i e^{i2\gamma_i} e^{-i\tau} + 2i\dot{a}^* P_i e^{i2\gamma_i} e^{-i\tau}) \qquad (5.19.a)$$

$$\dot{\omega}_3 = -\epsilon \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{I_3} \ddot{P}_i \qquad (5.19.b)$$

$$\ddot{P}_i + 2\delta_i \dot{P}_i + k_i^2 P_i = -\epsilon\dot{\omega}_3 + i\epsilon(a^2 e^{-i2\gamma_i} e^{i\tau} - a^{*2} e^{i2\gamma_i} e^{-i\tau}) + \frac{f_{oi}E'(1)}{2\mu_i \ell_i^4} (e^{i(\tau+\tau_i)} + e^{-i(\tau+\tau_i)}) \qquad (5.19.c)$$

$$\dot{\tau} = \omega_3 . \qquad (5.19.d)$$

Since we suppose that the effect of solar heating is so small that the induced vibrations of the appendages are small, we can write P_i and f_{oi} in the form

$$\left.\begin{aligned} P_i &= \epsilon \hat{P}_i \\ f_{oi} &= \epsilon \hat{f}_{oi} \end{aligned}\right\} \qquad (5.20)$$

Substituting these into Eqs. (5.19), we obtain

$$\dot{a} = i\left(\alpha - \frac{\omega_3}{2}\right) a + \epsilon^2 \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{2I} \left(2ia\dot{\hat{P}}_i + 2ia^* \dot{\hat{P}}_i e^{i2\gamma_i} e^{-i\tau} - \omega_3 a^* \hat{P}_i e^{i2\gamma_i} e^{-i\tau} + 2i\dot{a}^* \hat{P}_i e^{i2\gamma_i} e^{-i\tau}\right) \qquad (5.21.a)$$

$$\dot{\omega}_3 = -\epsilon^2 \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{I_3} \ddot{\hat{P}}_i \qquad (5.21.b)$$

$$\ddot{\hat{P}}_i + 2\delta_i \dot{\hat{P}}_i + k_i^2 \hat{P}_i = -\dot{\omega}_3 + i\left(a^2 e^{-i2\gamma_i} e^{i\tau} - a^{*2} e^{i2\gamma_i} e^{-i\tau}\right) + \frac{f_{oi} E'(1)}{2\mu_i \ell_i^4} \left(e^{i(\tau + \tau_i)} + e^{-i(\tau + \tau_i)}\right) \qquad (5.21.c)$$

$$\dot{\tau} = \omega_3 \qquad (5.21.d)$$

Let us assume that the spin velocity of the spacecraft is approximately equal to twice the angular velocity of the nutational body motion, i.e.,

$$2\alpha \cong \omega_3 \,. \qquad (5.22)$$

This condition means that

$$\frac{I_3}{I} \cong \frac{3}{2} \qquad (5.22)'$$

Then, we can conclude from Eqs. (5.21) that the variables (a, ω_3) are slowly varying functions because $(\dot{a}, \dot{\omega}_3) \sim O(\epsilon)$, while the variables (P_i, τ) vary relatively rapidly. Hence, approximate solutions of Eqs. (5.21) can be obtained by the method of averaging applied to a system containing both slow and rapid motions.

Briefly, the method of averaging applied to a system with rapid and slow motions is as follows : [37] Let us write such system in the following form :

$$\dot{x} = \epsilon X(x, y, t, \epsilon) \tag{5.23.a}$$

$$\dot{y} = Y(x, y, t, \epsilon) \tag{5.23.b}$$

where $x = (x_1, \ldots, x_n)$, $X = (X_1 \ldots, X_n)$ and $y = (y_1 \ldots, y_m)$, $Y = (Y_1 \ldots, Y_m)$ are n-and m- dimensional vector functions, respectively. The variables x_i are slowly varying since $\dot{x}_i \sim \epsilon$, while the variables y_i are rapidly varying since $\dot{y}_i \sim 1$. Together with the system (5.23) we shall also consider the degenerate system :

$$x = \text{const} \tag{5.24.a}$$

$$\dot{y} = Y_o(x, y, t) = Y(x, y, t, 0) \tag{5.24.b}$$

When a solution is given by the form

$$\left.\begin{aligned} x &= \bar{x} + \sum_{n=1}^{\infty} \epsilon^n U_n(x, y, t,) \\ y &= \bar{y} + \sum_{n=1}^{\infty} \epsilon^n V_n(x, y, t,) \end{aligned}\right\} \tag{5.25}$$

where U_i, V_i are n- and m- dimensional vector functions, respectively, the equation of the first approximation to x is obtained in the form

$$\dot{\bar{x}} = \epsilon \bar{X}_o(\bar{x}) \tag{5.26}$$

$$\bar{X}_o = \lim_{T\to\infty} \frac{1}{T}\int_o^T X(\bar{x},\ \zeta(\bar{x}, t), t, 0)\, dt \tag{5.27}$$

where $\zeta(\bar{x}, t)$ is an integral curve of the system (5.24.b). The solution of Eq. (5.26) is expected to approximate, with an error ϵ, to the solution of the original system (5.23.a) on an interval $t \sim \epsilon^{-1}$.

5.3 The approximate Solutions and Their Stability

The method as outlined above can be applied to Eqs. (5.21) to obtain the first approximate solutions. Equations corresponding to Eqs. (5.24) are given by

$$a = \bar{a} \tag{5.28.a}$$

$$\omega_3 = \bar{\omega}_3 \tag{5.28.b}$$

$$\ddot{\hat{P}}_i + 2\delta_i\dot{\hat{P}}_i + \bar{k}_i^2\hat{P}_i = i\,(\bar{a}^2 e^{-i2\gamma_i} e^{i\tau} - \bar{a}^{*2} e^{i2\gamma_i} e^{-i\tau}) + \frac{\hat{f}_{oi} E'(1)}{2\mu_i \ell_i^4}(e^{i(\tau+\tau_i)} + e^{-i(\tau+\tau_i)}) \tag{5.28.c}$$

$$\dot{\tau} = \bar{\omega}_3 \tag{5.28.d}$$

where $\bar{k}_i^2 = \dfrac{\lambda^4 B_i}{\mu_i \ell_i^4} + (\beta - 1)\,\bar{\omega}_3^2$.

From Eq. (5.28.d)

$$\tau = \bar{\omega}_3 t. \tag{5.29}$$

The steady-state solutions of Eqs. (5.28.c) are given by

$$\hat{P}_i = \frac{i\bar{a}^2 e^{-i2\gamma_i} e^{i\bar{\omega}_3 t}}{\bar{k}_i^2 + 2i\delta_i \bar{\omega}_3 - \bar{\omega}_3^2} + \frac{\hat{f}_{oi} E'(1)}{2\mu_i \ell_i^4} \frac{e^{i(\bar{\omega}_3 t + \tau_i)}}{\bar{k}_i^2 + 2i\delta_i \bar{\omega}_3 - \bar{\omega}_3^2}$$

$$+ \text{ complex conjugate part.} \tag{5.30}$$

Substituting Eqs. (5.29), (5.30) into Eqs. (5.21.a), (5.21.b) and averaging as in Eq. (5.27), we obtain an averaged system of equations as follows :

$$\dot{\bar{a}} = i\left(\bar{\alpha} - \frac{\bar{\omega}_3}{2}\right)\bar{a} - \epsilon^2 \sum_{i=1}^{N} \frac{\mu_i \ell_i^3}{2I} \left(\frac{3i\bar{\omega}_3 \bar{a}^2 \bar{a}^*}{\bar{k}_i^2 + 2i\delta_i \bar{\omega}_3 - \bar{\omega}_3^2} \right.$$

$$\left. + \frac{\hat{f}_{oi} E'(1)}{2\mu_i \ell_i^4} \frac{3\bar{\omega}_3 \bar{a}^* e^{i3\tau_i}}{\bar{k}_i^2 + 2i\delta_i \bar{\omega}_3 - \bar{\omega}_3^2} \right) \tag{5.31.a}$$

$$\dot{\bar{\omega}}_3 = 0 \tag{5.31.b}$$

where

$$\bar{\alpha} = \left(\frac{I_3}{I} - 1\right)\bar{\omega}_3 .$$

Eq. (5.31.b) yields

$$\bar{\omega}_3 = \omega_o . \tag{5.32}$$

From this equation, Eq. (5.29) becomes

$$\tau = \omega_o t. \tag{5.33}$$

Putting

$$a = (x + iy)\omega_o \tag{5.34}$$

and introducing a new independent variable

$$\tau = \omega_o t$$

we obtain from Eq. (5.31.a)

$$\left.\begin{aligned} x' &= X(x,\ y) \\ y' &= Y(x,\ y) \end{aligned}\right\} \qquad (5.35)$$

where

$$X(x,y) = h_o y + h_{1r} x + h_{1i} y + r^2 (h_{2r} x - h_{2i} y)$$

$$Y(x,y) = -h_o x + h_{1i} x - h_{1r} y + r^2 (h_{2r} y + h_{2i} x)$$

$$h_o = \left(\frac{1}{2} - \frac{\alpha_o}{\omega_o}\right)$$

$$h_{1r} = -\ \epsilon^2 \sum_{i=1}^{N} \frac{\hat{f}_{oi} E'(1)}{4\ell_i I} \frac{(k_{io}{}^2 - \omega_o{}^2)\, C3\tau_i + 2\delta_i \omega_o S3\tau_i}{(k_{io}{}^2 - \omega_o{}^2)^2 + (2\delta_i \omega_o)^2}$$

$$h_{1i} = -\ \epsilon^2 \sum_{i=1}^{N} \frac{\hat{f}_{oi} E'(1)}{4\ell_i I} \frac{(k_{io}{}^2 - \omega_o{}^2)\, S3\tau_i - 2\delta_i \omega_o C3\tau_i}{(k_{io}{}^2 - \omega_o{}^2)^2 + (2\delta_i \omega_o)^2}$$

$$h_{2r} = -\ \epsilon^2 \sum_{i=1}^{N} \frac{3\mu_i \ell_i^3\, \omega_o^2}{2I} \frac{2\delta_i \omega_o}{(k_{io}{}^2 - \omega_o{}^2)^2 + (2\delta_i \omega_o)^2}$$

$$h_{2i} = -\ \epsilon^2 \sum_{i=1}^{N} \frac{3\mu_i \ell_i^3\, \omega_o^2}{2I} \frac{(k_{io}{}^2 - \omega_o{}^2)}{(k_{io}{}^2 - \omega_o{}^2)^2 + (2\delta_i \omega_o)^2}$$

$$\alpha_o = \left(\frac{I_3}{I} - 1\right)\omega_o,\ k_{io}{}^2 = \frac{\lambda^4 B_i}{\mu_i \ell_i^4} + (\beta-1)\omega_o{}^2\ , \qquad r^2 = x^2 + y^2$$

The primes denote the differentiation with respect to τ. It may be noted, from Eqs. (5.18). (5.34), the quantity r is proportional to the amplitude of a nutational body motion, $\sqrt{\omega_1{}^2 + \omega_2{}^2}\Big/\omega_o$. By the use of the method of averaging, the system of Eqs. (5.21) has been reduced to the system of Eqs. (5.35). This system of equations is that of two ordinary differential equations of the first order, so that we can successfully apply the topological method to obtain the over-all views of the motion.

Let us now consider in more detail the steady state where the quantities x and y in Eqs. (5.35) are constant :

$$\left.\begin{aligned} X(x,\ y) &= 0 \\ Y(x,\ y) &= 0 \end{aligned}\right\} \tag{5.36}$$

Substitution of these conditions into Eqs. (5.35) leads to the determination of the steady state amplitude of $r_o(=\sqrt{x_o^2 + y_o^2})$ as follows :

$$r_o^2 = 0 \tag{5.37.a}$$

$$r_o^2 = \frac{h_o h_{2i}}{(h_{2r}^2 + h_{2i}^2)} \pm [(h_o h_{2i})^2 + (h_{1r}^2 + h_{1i}^2 - h_o^2) \times (h_{2r}^2 + h_{2i}^2)]^{\frac{1}{2}} \tag{5.37.b}$$

and the components x_o, y_o of the amplitude r_o are

$$\left.\begin{aligned} x_o^2 &= \frac{r_o^2}{1 + (\frac{P}{Q})^2} \\ y_o^2 &= \frac{r_o^2}{1 + (\frac{Q}{P})^2} \end{aligned}\right. \tag{5.38}$$

where

$$P = h_{1r} + r_o^2 h_{2r}$$

$$Q = h_o + h_{1i} - r_o^2 h_{2i} .$$

As an example, a symmetrical spacecraft having three equal appendages is investigated. The appendages make an angle of 120° with each other. Figure 5.2 shows the relationship between h_o and r_o^2 in the case where $h_{1r} = 0.110$,

$h_{1i} = 0.441,\ h_{2r} = 1.51,\ h_{2i} = -3.77$

Then, let us investigate the stability of the steady state solutions (5.35). To establish necessary and sufficient conditions for the stability of these solutions, we must construct the variational equations about these solutions. The variational coordinates are characterized by the symbols δx and δy. From Eqs. (5.35) we obtain the variational equations as follows :

$$\left.\begin{aligned} \delta x' &= a_1\ \delta x + a_2\ \delta y \\ \delta y' &= b_1\ \delta x + b_2\ \delta y \end{aligned}\right\} \tag{5.39}$$

where

$$a_1 = \left(\frac{\partial X}{\partial x}\right)_{\substack{x=x_o \\ y=y_o}} = h_{1r} + r_o^2 h_{2r} + 2h_{2r} x_o^2 - 2x_o y_o h_{2i}$$

$$a_2 = \left(\frac{\partial X}{\partial y}\right)_{\substack{x=x_o \\ y=y_o}} = (h_o + h_{1i}) - r_o^2 h_{2i} - 2h_{2i} y_o^2 + 2x_o y_o h_{2r}$$

$$b_1 = \left(\frac{\partial Y}{\partial x}\right)_{\substack{x=x_o \\ y=y_o}} = (-h_o + h_{1i}) + r_o^2 h_{2i} + 2h_{2i} x_o^2 + 2x_o y_o h_{2r}$$

$$b_2 = \left(\frac{\partial Y}{\partial y}\right)_{\substack{x=x_o \\ y=y_o}} = -h_{1r} + r_o^2 h_{2r} + 2h_{2r} y_o^2 + 2x_o y_o h_{2i}$$

The characteristic equation of the system defined by Eq. (5.39) is

$$\lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1) = 0 \tag{5.40}$$

First, we shall consider the stability of the solution (5.37.a). The characteristic equation corresponding to this solution becomes

$$\lambda^2 - (h_{1r}^2 + h_{1i}^2 - h_o^2) = 0 \tag{5.41}$$

As the stability criterion, the roots of the characteristic equation must not have a positive real part, we have

$$h_{1r}^2 + h_{1i}^2 - h_o^2 \leqq 0 . \tag{5.42}$$

If the condition (5.42) is saitsfied, the characteristic roots are purely imaginary. Since the condition that the characteristic roots are purely imaginary is not sufficient for the stability of the solution, the stability condition must be examined in more detail from the consideration of the type of the singularity at the origin of the system of Eq. (5.35), which is correlated with the solution (5.37.a). By virture of the procedure due to Poincare [38] it is clarified that

if $h_{2r} < 0$, the singularity is a stable focus,

if $h_{2r} > 0$, the singularity is an unstable focus,

if $h_{2r} = 0$, the singularity is a center.

The stability conditions are, therefore, set up as follows :

$$h_{2r} \leq 0 \tag{5.43.a}$$

$$h_{1r}^2 + h_{1i}^2 - h_o^2 \leq 0. \tag{5.43.b}$$

Next, we shall consider the stability of the solution (5.37.b). The characteristic equation corresponding to this solution is given by

$$\lambda^2 - 4h_{2r} r_o^2 \lambda + 4r_o^2 \; [\; h_{2r}^2 r_o^2 - (h_o - r_o^2 h_{2i}) h_{2i} \;] = 0. \tag{5.44}$$

Then, the stability condition is derived by the Routh-Hurwitz criterion :

$$h_{2r} < 0 \tag{5.45.a}$$

$$(h_{2r}^2 + h_{2i}^2)\, r_o^2 - h_o h_{2i} > 0. \tag{5.45.b}$$

It may be noted that the conditions (5.43.a), (5.45.a) are always fullfilled, since we are concerned with a damped mechanical system. Let the stability conditions (5.43.b), (5.45.b) be represented in Fig. 5.2. In Fig. 5.2, the dashed parts of the response curves are unstable and the solid lines of the response curves are stable. These peculiar conditions for stability of nonlinear external resonance cause the appearance of the so-called jumping phenomenon [38,39]. In fig. 5.2, with increasing h_o, it is observed that the nutational body motion starts abruptly with a finite amplitude for $h_o = h_{o1}$ and decreases smoothly for $h_o > h_{o1}$. On the contrary, for decreasing h_o it is observed that the phenomenon is different; namely, for $h_o = h_{o1}$ the nutational body motion does not disappear and it jumps down and disappears at $h_o = h_{o2}$:

5.4 Domains of attraction

We shall here investigate the relationship between the initial conditions and the resulting steady state solutions of a system governed by Eqs. (5.35). For this purpose it is useful to investigate the intregral curves of the following equation derived from Eqs. (5.35), i.e.,

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} . \tag{5.46}$$

A singular point, for which $X(x, y) = 0$ and $Y(x, y) = 0$, corresponds to a steady state solution of Eqs. (5.35). Let us trace the integral curves for certain typical cases on the x, y plane. The special case considered here are characterized by the following values of the system parameters :

Case 1 : $h_o = -0.25$, $h_{1r} = 0.110$, $h_{1i} = -0.0441$, $h_{2i} = -1.51$, $h_{2r} = -3.77$

Case 2 : $h_o = 0.0$ 〃 〃 〃 〃

With the aid of the numerical integration of Eq. (5.46), the integral curves for these cases are drawn in Fig. 5.3.a and 5.3.b, respectively. The singularities in Fig. 5. 3. a and 5. 3. b are listed in Table 5.1. The integral curve which tends to a saddle point with increasing the time t is a separatrix which divides the coordinate plane into two regions, where any initial condition results in passage to a particular class of steady state solutions. From these figures the relationship between initial conditions and the resulting steady state solutions is easy to understand : In Fig. 5.3.b, a nutational body motion started with any initial conditions in the shaded region tends ultimately to the singularity of point 2, whereas a nutational body motion started from the unshaded region tends to the singularity of point 1.

Finally, let us investigate conservative systems. Although these systems are somewhat unrealistic, the characteristics of the integral curves are interesting. Putting $\delta_i = 0$ in Eqs. (5.35), we obtain

$$\frac{dy}{dx} = \frac{Y_o(x, y)}{X_o(x, y)} \tag{5.47}$$

where

$$\left.\begin{aligned} X_o(x, y) &= h_o y + h_{1r} x + h_{1i} y - r^2 y h_{2i} \\ Y_o(x, y) &= -h_o x + h_{1i} x - h_{1r} y + r^2 x h_{2i} \end{aligned}\right\} \tag{5.48}$$

from which we obtain

$$X_o(x, y)dy - Y_o(x, y)\, dx = 0. \tag{5.49}$$

Since from Eqs. (5.48)

$$\frac{\partial X_o}{\partial x} + \frac{\partial Y_o}{\partial y} = 0, \tag{5.50}$$

Eq. (5.47) becomes an exact differential equation, and the complete integral is given by

$$c = \frac{h_o}{2}(x^2 + y^2) - \frac{h_{1i}}{2}(x^2 - y^2) - h_{2i}\left(\frac{x}{2} + \frac{y}{2}\right)^2 + h_{1r} xy \tag{5.51}$$

where c is a constant of integration. The integral curves of Eq. (5.49) are readily obtained by plotting Eq. (5.51). Figure 5.4 shows the integral curves of Eq. (5.49), where the system parameter values are given by $h_0 = -0.25$, $h_{1r} = 0.128$, $h_{1i} = 0.0$, $h_{2r} = 0.0$, $h_{2i} = -4.37$. It is noted that, in a conservative system, each integral curve forms a closed trajectory and does not tend to a stable singularity. This means that the amplitude of the nutational body motion exhibits slowly periodic changes.

5.5 Conclusions

It has been demonstrated that a spacecraft with flexible appendages may exhibit thermally induced nutational body motions, as illustrated in Fig. 5.2. The amplitude of the nutational body motion is determined analytically and the stability of the motion is examined in detail.

This phenomenon is considered an external resonance pehnomenon of a nonlinear mechanical system with several degrees of freedom. In the preceeding work, the use of the method of averaging is seen to reduce the fundamental equations to a system of two ordinary nonlinear differential equations of the first order, so that the general nature of the mechanical system is easily obtained.

Table 5.1 Singular Points in Figs. 5. 3. a and 5.3.b

Singular point	x_o	y_o	λ	Classification
Fig. 5. 3. a				
1	0	0	$\pm i\,0.221$	Stable focus
2	0.259	−0.0909	$-0.228 \pm i\,0.198$	"
3	−0.259	0.0909	"	"
4	0.186	0.0653	-0.117 ± 0.246	Saddle
5	−0.186	−0.0653	"	"
Fig. 5. 3. b				
1	0.121	−0.121	$-0.883 \pm i\,0.220$	Stable focus
2	−0.121	0.121	"	"
3	0	0	± 0.119	Saddle

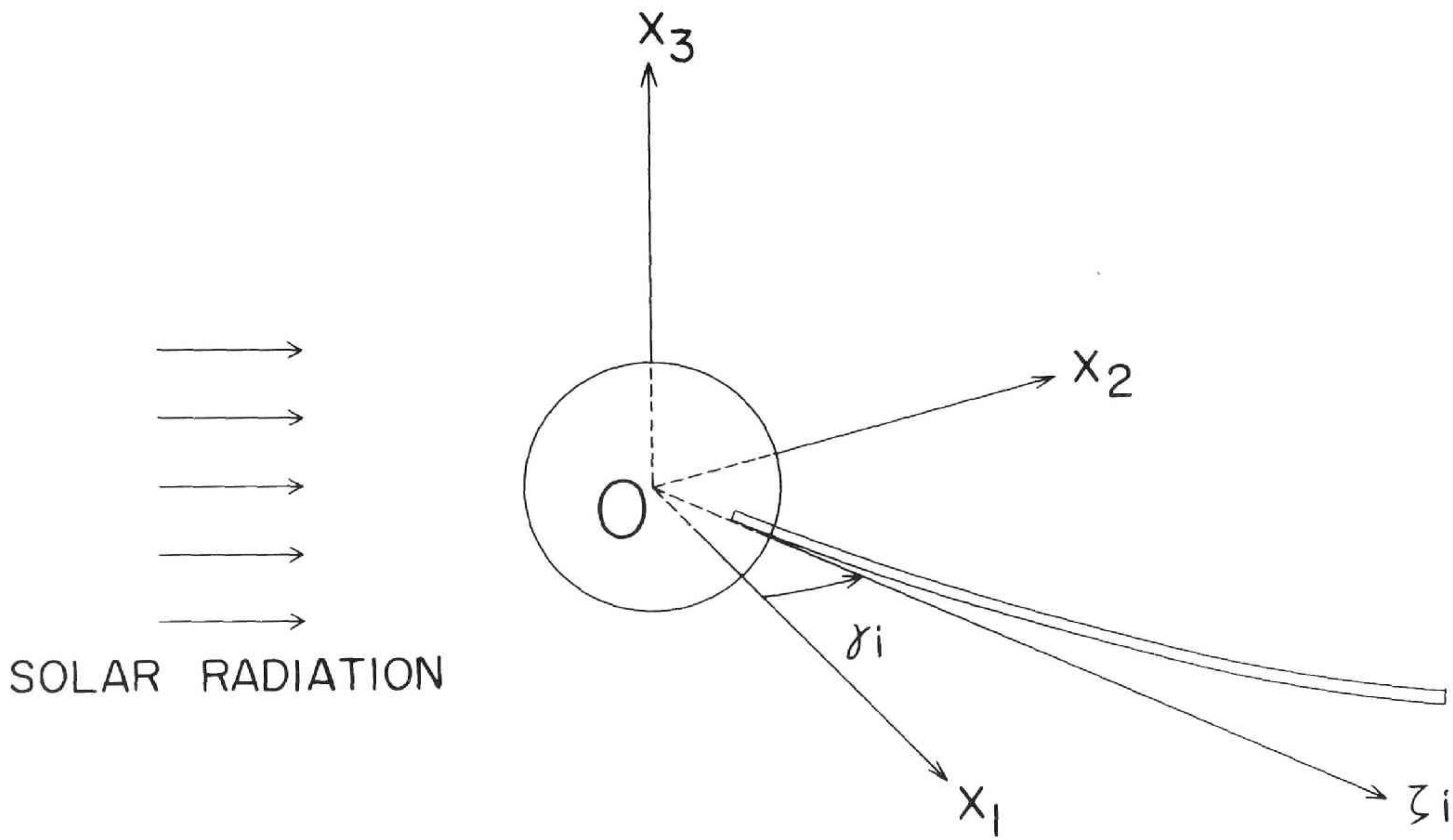

Fig. 5.1 **Spacecraft configuration**

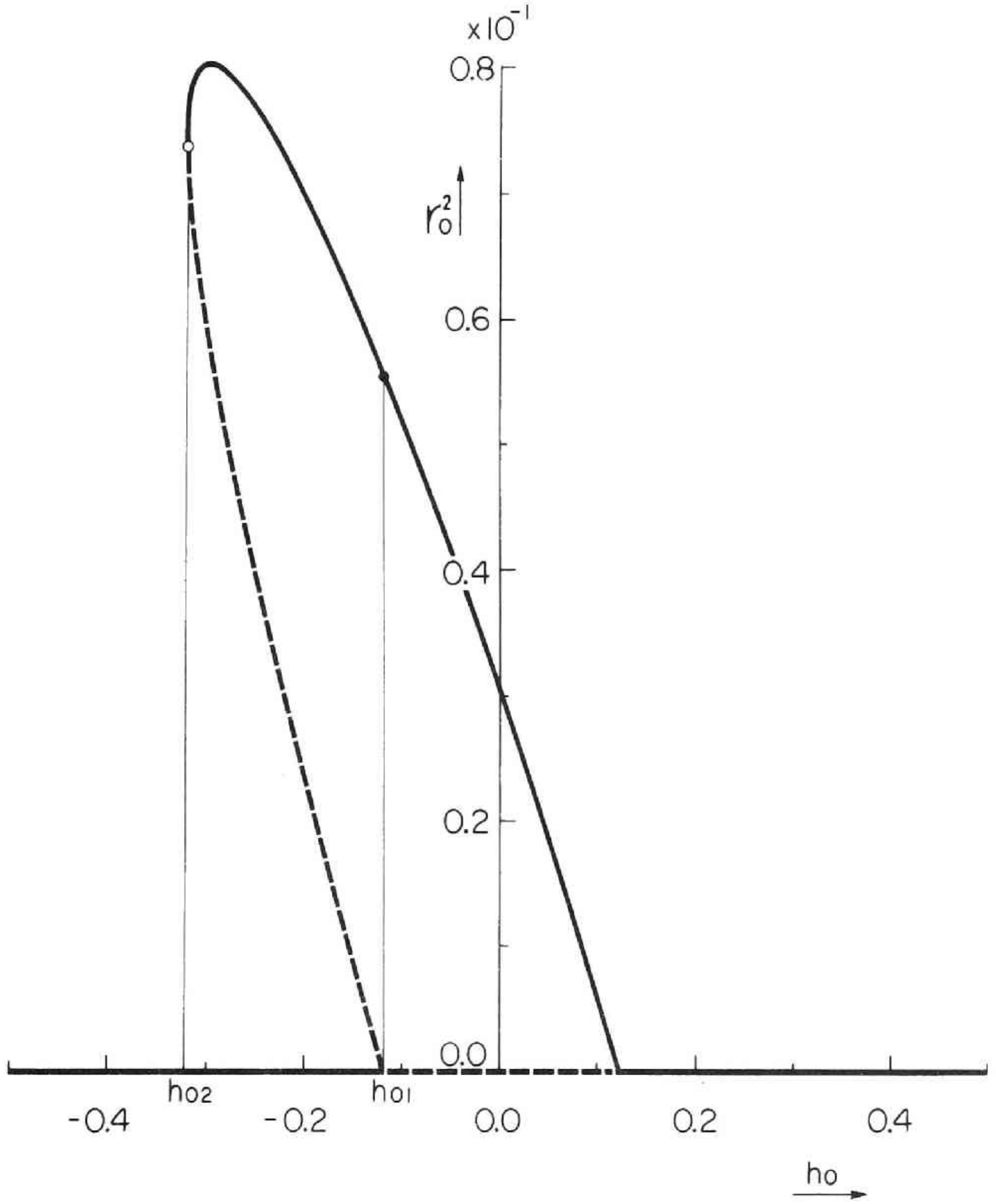

Fig. 5.2 Frequency responce curves of nutational body motion

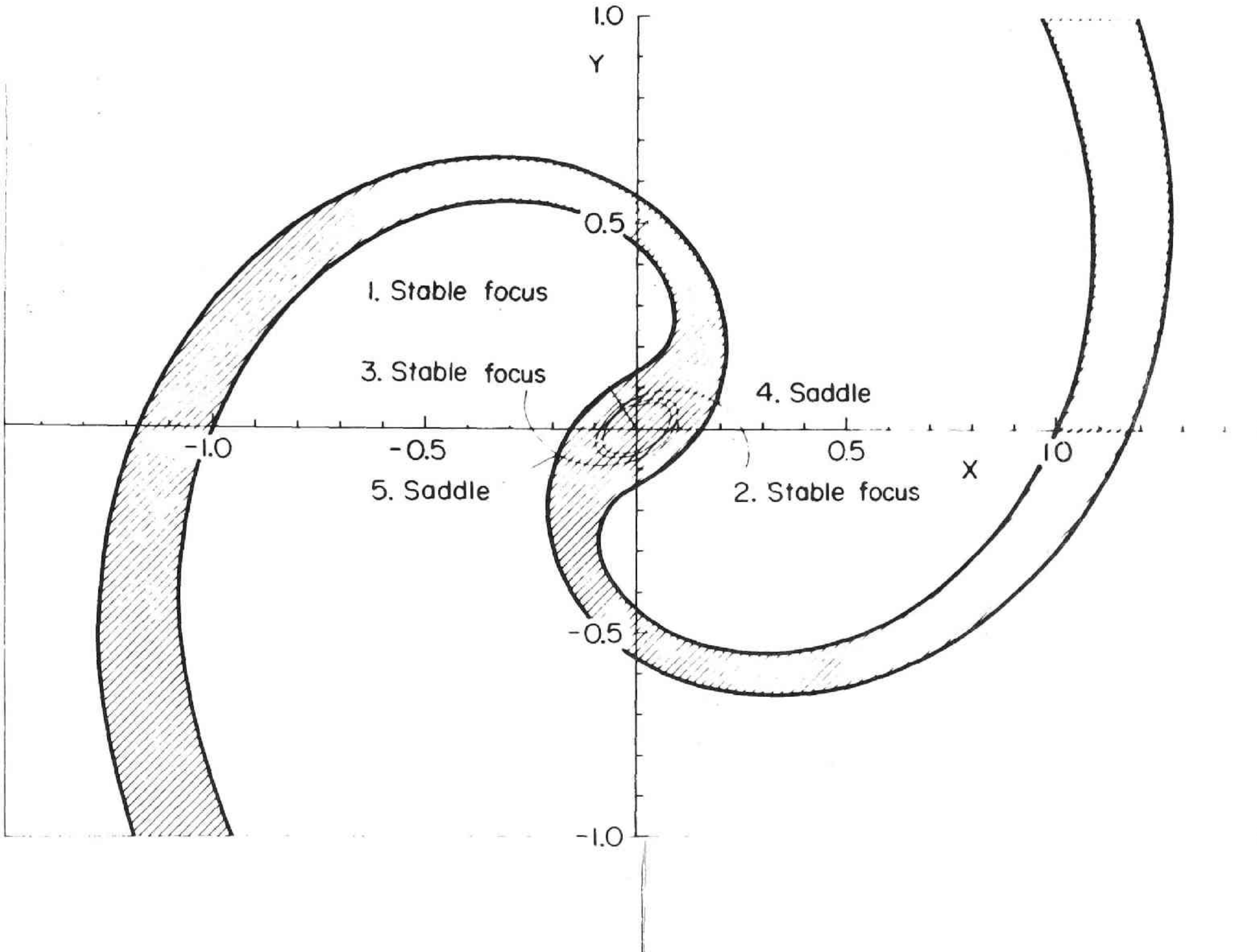

Fig. 5.3.a Integral curves of Eq. (5.46) in the xy plane (case 1)

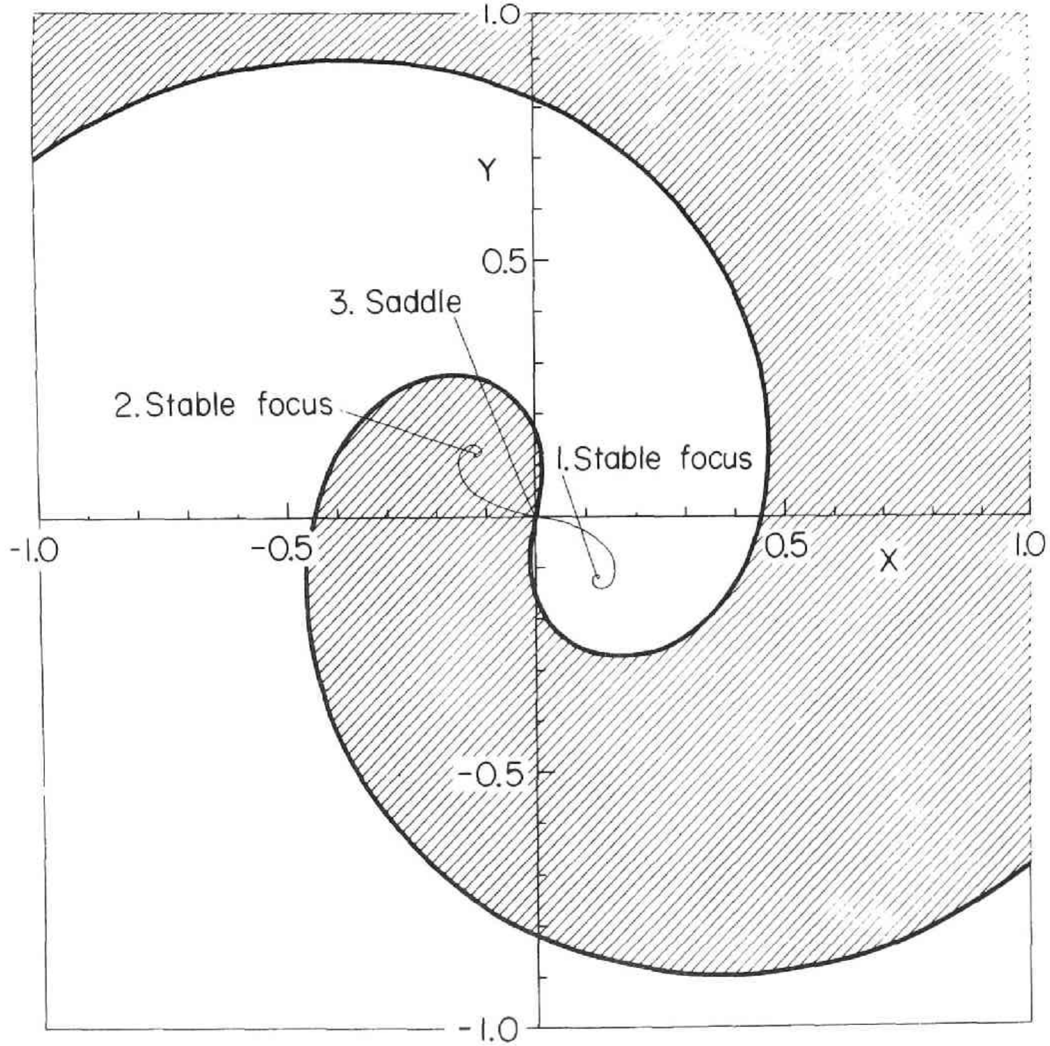

Fig. 5.3.b **Integral curves of Eq. (5.46) in the xy plane (case 2)**

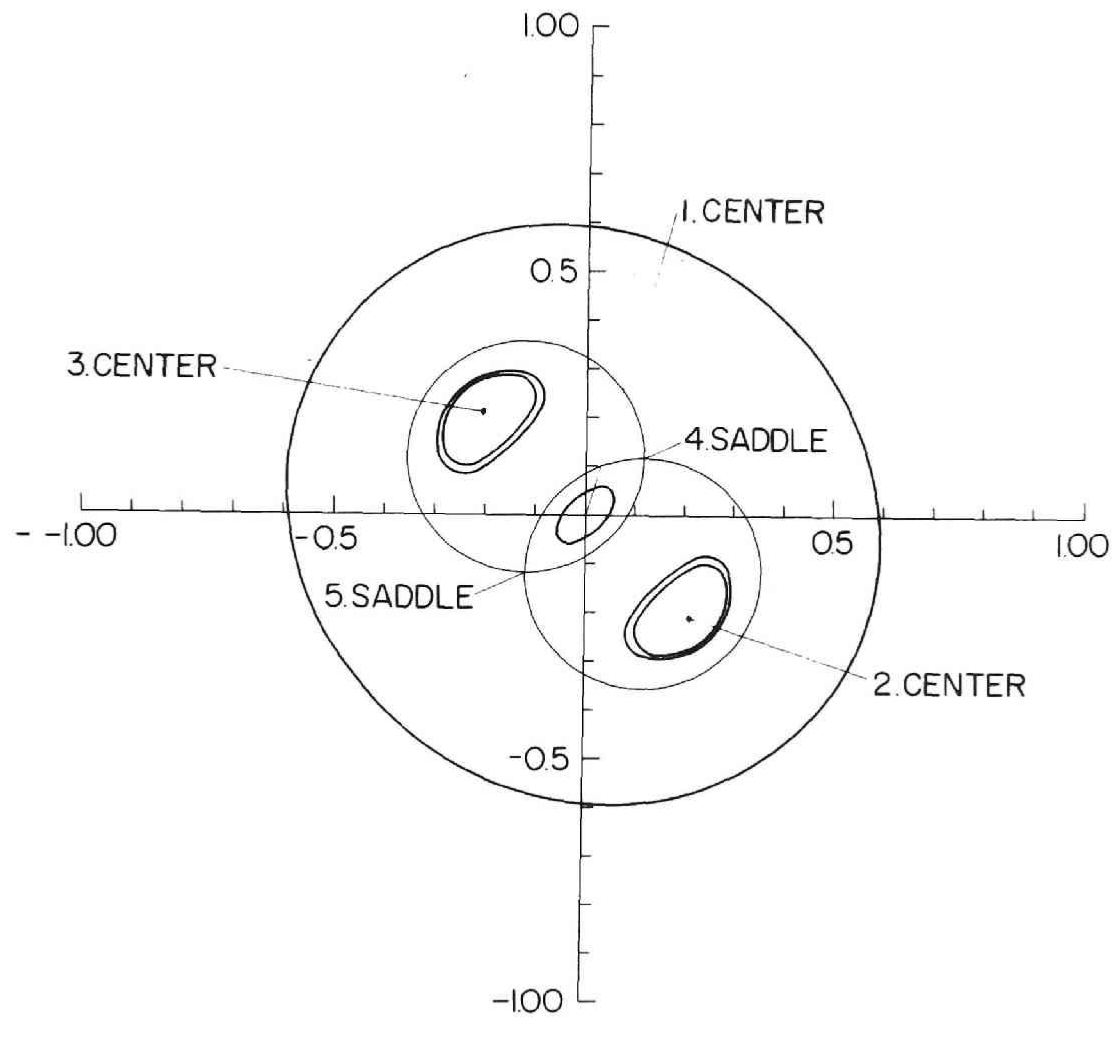

Fig. 5.4 Integral curves of Eq. (5.49) in the xy plane

APPENDIX

KINETIC ENERGY AND ANGULAR MOMENTUM EXPRESSIONS OF A SPACECRAFT WITH HAVING FLEXIBLE APPENDAGES

Let us consider a spacecraft composed of a central rigid body B and flexible appendages A_i as shown in Fig.A.1. As illustrated in Fig.A.1, P designates the mass center of the total configuration; O is selected so as to be coincident with P when the vehicle is in some nominally undeformed configuration; N designates the point in inertia space occupied at the time t=0 by the point P. Define the position vectors relating the points P, O and N as follows: $\underline{x}$ is the vector from N to the point P, $\underline{c}$ is the vector from P to the point O. Let us denote by $\underline{\rho}$ a vector from O to an element of mass dm in B. Let us denote by $\underline{r}_i$ a position vector relative to the point O of an element of mass dm_i in A_i in the nominally undeformed state and by $\underline{w}_i$ an elastic displacement vector of the element dm_i. Orthogonal unit vectors $\underline{b}_1, \underline{b}_2, \underline{b}_3$ are fixed along the principal axes of the system in the undeformed configuration. For an appendage i, an orthonormal set $\underline{a}_{i1}, \underline{a}_{i2}, \underline{a}_{i3}$ is defined so that the principal axes of a body A_i in the undeformed state are assumed parallel to the $\underline{a}_i$ set. The two vector bases are related by the direction cosine matrix C_i, i.e.,

$$[\, \underline{a}_i] = C_i \, [\, \underline{b} \,] \tag{A.1}$$

where

$$[\, \underline{a}_i] = \begin{pmatrix} \underline{a}_{i1} \\ \underline{a}_{i2} \\ \underline{a}_{i3} \end{pmatrix} \qquad [\, \underline{b}_1] = \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{pmatrix}$$

The system kinetic energy T is given by

$$2T = \int_B (\dot{\underline{x}} + \dot{\underline{c}} + \dot{\underline{\rho}}) \cdot (\dot{\underline{x}} + \dot{\underline{c}} + \dot{\underline{\rho}}) \, dm$$

$$+ \sum_{i=1}^{N} \int_{A_i} (\dot{\underline{x}} + \dot{\underline{c}} + \dot{\underline{r}}_i + \underline{w}_i) \cdot (\dot{\underline{x}} + \dot{\underline{c}} + \dot{\underline{w}}_i) \, dm_i \tag{A.2}$$

where $\int_B dm$ and $\int_{A_i} dm_i$ denote that the integrations are carried out over the body B and an appendage A_i, respectively. The dot over a vector denotes the time differentiation of that vector with respect to an inertia frame. The mass center definition requires that

$$\int_B (\underline{c} + \underline{\rho}) \, dm + \sum_{i=1}^{N} \int_{A_i} (\underline{c} + \underline{r}_i + \underline{w}_i) \, dm_i = 0 \ , \tag{A.3}$$

or

$$M\underline{c} = -\left[\int_B \underline{\rho} \, dm + \sum_{i=1}^{N} \int_{A_i} (\underline{r}_i + \underline{w}_i) \, dm_i \right] \tag{A.3$'$}$$

where M is the total mass of the vehicle. From the definition of the point O, it follows that

$$\int_B \underline{\rho} \, dm + \sum_{i=1}^{N} \int_{A_i} \underline{r}_i \, dm_i = 0 \ .$$

Hence, Eq. (A.3)$'$ reduces to

$$M\underline{c} = -\sum_{i=1}^{N} \int_{A_i} \underline{w}_i \, dm_i \ . \tag{A.3$''$}$$

By virture of the definition of the mass center given by Eq. (A.3), the kinetic energy of the system can be written as

$$2T = M\dot{\underline{x}} \cdot \dot{\underline{x}} - M\dot{\underline{c}} \cdot \dot{\underline{c}} + \int_B \dot{\underline{\rho}} \cdot \dot{\underline{\rho}} \, dm$$

$$+\sum_{i=1}^{N}\int_{A_i}(\dot{\underline{r}}_i+\dot{\underline{w}}_i)(\dot{\underline{r}}_i+\dot{\underline{w}}_i)\,dm_i\,. \tag{A.4}$$

The first term in Eq. (A.4) is the kinetic energy of the translational motion of the mass center of the system. This will be ignored in what follows because it will be assumed that the motion of the mass center is not affected by the motion relative to the mass center. Now, denote by $\underline{\omega}$ the inertia angular velocity vector of the vector basis [$\underline{b}$] . Then, the inertia space time derivative $\dot{\underline{a}}$ ($\underline{a}$ is any vector) is given by

$$\dot{\underline{a}}=\mathring{\underline{a}}+\underline{\omega}\times\underline{a} \tag{A.5}$$

where $\mathring{\underline{a}}$ implies the time differentiation of the vector $\underline{a}$ with respect to the vector basis [$\underline{b}$] . Using Eq. (A.5), Eq.(A.4) reduces to

$$2T=-M\mathring{\underline{c}}\,\mathring{\underline{c}}+2\mathring{\underline{c}}(\underline{\omega}\times\underline{c})+(\underline{\omega}\times\underline{c})(\underline{\omega}\times\underline{c})$$

$$+\int_B(\underline{\omega}\times\underline{\rho})(\underline{\omega}\times\underline{\rho})\,dm+\sum_{i=1}^{N}\int_{A_i}\Big\{\mathring{\underline{w}}_i\,\mathring{\underline{w}}_i$$

$$+2\mathring{\underline{w}}_i\,[\,\underline{\omega}\times(\underline{r}_i+\underline{w}_i)]+[\,\underline{\omega}\times(\underline{w}_i+\underline{r}_i)\,][\,\underline{\omega}\times(\underline{w}_i+\underline{r}_i)\,]\Big\}dm_i\,. \tag{A.6}$$

Let us write the system kinetic energy in matrix notation. For this purpose, a 3 by 1 matrix a is defined for any vector $\underline{a}$ in terms of the vector basis [$\underline{b}$] so that

$$\underline{a}=[\,\underline{b}\,]^T a, \qquad a=[\,a_1,a_2,a_3\,]^T \tag{A.7}$$

where the superscript T denotes the transpose of a vector array or a matrix. Furthermore, for a vector $\underline{a}$, a 3 by 3 skew-symmetric matrix $\tilde{a}$ is constructed according to the following rule

$$\tilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{A.8}$$

Then, the kinetic energy of the system is written in matrix notation with respect to the vector basis $[\underset{\sim}{b}]$ as follows :

$$2T = \omega^T I_o \omega + \sum_{i=1}^{N} \int_{A_i} \Big\{ \dot{w}_i^T \dot{w}_i - 2 (\dot{w}_i^T \tilde{w}_i + \dot{w}_i^T \tilde{r}_i)\, \omega$$

$$+ \omega^T [(\tilde{r}_i + \tilde{w}_i)^T (\tilde{r}_i + \tilde{w}_i)] \omega \Big\} dm_i \tag{A.9}$$

$$- M (\dot{c}^T \dot{c} - 2\dot{c}^T \tilde{c} + \omega^T \tilde{c}^T \tilde{c}\, \omega)$$

where I_o is the inertia matrix of the total vehicle about O in the undeformed configuration. The matrix I_o is diagonal since $\underset{\sim}{b}_1$, $\underset{\sim}{b}_2$, $\underset{\sim}{b}_3$ are assumed parallel to the principal axes of the undeformed total configuration, i.e.,

$$I_o = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

The matrices ω, c, w_i, r_i may be written in expanded forms as

$$\left.\begin{aligned} \omega &= [\, \omega_1, \omega_2, \omega_3 \,]^T \\ c &= [\, c_1, c_2, c_3 \,]^T \\ r_i &= [\, r_{i1}, r_{i2}, r_{i3} \,]^T \\ w_i &= [\, w_{i1}, w_{i2}, w_{i3} \,]^T . \end{aligned}\right\} \tag{A.10}$$

Then, by expanding the matrices, the system kinetic energy T is written as

$$
\begin{aligned}
2T = {} & I_1\omega_1^{\,2} + I_2\omega_2^{\,2} + I_3\omega_3^{\,2} + \sum_{i=1}^{N}\int_{A_i}\Big\{(\dot w_{i1}^{\,2} + \dot w_{i2}^{\,2} + \dot w_{i3}^{\,2}) \\
& -2\omega_1(\dot w_{i2}w_{i3} - \dot w_{i3}w_{i2} + \dot w_{i2}r_{i3} - \dot w_{i3}r_{i2}) - 2\omega_2(\dot w_{i3}w_{i1} - \dot w_{i1}w_{i3} \\
& + \dot w_{i3}r_{i1} - \dot w_{i1}r_{i3}) - 2\omega_3(\dot w_{i1}w_{i2} - \dot w_{i2}w_{i1} + \dot w_{i1}r_{i2} - \dot w_{i2}r_{i1}) \\
& + [\,\omega_1\,\omega_2\,\omega_3\,]\,\Big[\begin{pmatrix}
(w_{i3}+r_{i3})^2 + (w_{i2}+r_{i2})^2 & -(w_{i2}+r_{i2})(w_{i1}+r_{i1}) & -(w_{i1}+r_{i1})(w_{i3}+r_{i3}) \\
-(w_{i1}+r_{i1})(w_{i2}+r_{i2}) & (w_{i3}+r_{i3})^2 + (w_{i1}+r_{i1})^2 & -(w_{i3}+r_{i3})(w_{i2}+r_{i2}) \\
-(w_{i1}+r_{i1})(w_{i3}+r_{i3}) & -(w_{i2}+r_{i2})(w_{i3}+r_{i3}) & (w_{i2}+r_{i2})^2 + (w_{i1}+r_{i1})^2
\end{pmatrix}
\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\Big]\Big\}\,dm_i \\
& -M\,[\,(\dot c_1^{\,2} + \dot c_2^{\,2} + \dot c_3^{\,2}) - 2\omega_1(\dot c_2 c_3 - \dot c_3 c_2) - 2\omega_2(\dot c_3 c_1 - \dot c_1 c_3) \\
& -2\omega_3(\dot c_1 c_2 - \dot c_2 c_1) \\
& + [\,\omega_1\,\omega_2\,\omega_3\,]\begin{pmatrix}
c_3^{\,2} + c_2^{\,2} & -c_1c_2 & -c_3c_1 \\
-c_1c_2 & c_1^{\,2} + c_3^{\,2} & -c_2c_3 \\
-c_1c_3 & -c_2c_3 & c_2^{\,2} + c_1^{\,2}
\end{pmatrix}
\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\,].
\end{aligned}
\tag{A.11}
$$

The system angular momentum $\underline{L}$ is given by

$$
\underline{L} = \int_B (\underline{c}+\underline{\rho}) \times (\dot{\underline{c}}+\dot{\underline{\rho}})\,dm + \sum_{i=1}^{N}\int_{A_i} (\underline{c}+\underline{r}_i+\underline{w}_i) \times (\dot{\underline{c}}+\dot{\underline{r}}_i+\dot{\underline{w}}_i)\,dm_i\,.
\tag{A.12}
$$

By using the mass center definition, Eq. (A.12) reduces to

$$\underline{L} = -M\underline{c}\times\dot{\underline{c}} + \int_B \underline{\rho}\times\dot{\underline{\rho}}\,dm + \sum_{i=1}^{N}\int_{A_i} (\underline{r}_i + \underline{w}_i)\times(\dot{\underline{r}}_i + \dot{\underline{w}}_i)\,dm_i . \qquad (A.13)$$

Substitution of Eq. (A.5) into Eq. (A.13) leads to

$$\underline{L} = M\mathring{\underline{c}}\times\underline{c} + M(\underline{\omega}\times\underline{c})\times\underline{c} + \int_B \underline{\rho}\times(\underline{\omega}\times\underline{\rho})\,dm$$

$$+ \sum_{i=1}^{N}\int_{A_i} (\underline{w}_i + \underline{r}_i)\times\mathring{\underline{w}}_i + (\underline{w}_i + \underline{r}_i)\times[\,\underline{\omega}\times(\underline{w}_i + \underline{r}_i)\,]\,dm_i . \qquad (A.14)$$

The matrix representation of $\underline{L}$ with respect to the vector basis $[\,\underline{b}\,]$ is written as

$$\underline{L} = [\,\underline{b}\,]^T \Big\{ M\widetilde{\dot{c}}\,c + M\,\widetilde{c}\,\widetilde{c}\,\omega + I_o\,\omega + \sum_{i=1}^{N}\int_{A_i} [\,\widetilde{w}_i\dot{w}_i$$

$$+\widetilde{r}_i\dot{w}_i + (\,\widetilde{w}_i^{\,T} + \widetilde{r}_i^{\,T})\,(\,\widetilde{w}_i + \widetilde{r}_i)\,\omega\,]\,dm_i. \qquad (A.15)$$

By using (A.10), Eq. (A.15) is written in the form

$$\underline{L} = [\,\underline{b}\,]^T \left\{ M \begin{pmatrix} \dot{c}_2 c_3 - \dot{c}_3 c_2 \\ \dot{c}_3 c_1 - \dot{c}_1 c_3 \\ \dot{c}_1 c_2 - \dot{c}_2 c_1 \end{pmatrix} - M \begin{pmatrix} c_3^{\,2} + c_2^{\,2} & -c_2 c_1 & -c_3 c_1 \\ -c_1 c_2 & c_1^{\,2} + c_3^{\,2} & -c_3 c_2 \\ -c_1 c_3 & -c_2 c_3 & c_2^{\,2} + c_1^{\,2} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \right.$$

$$+ \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} + \sum_{i=1}^{N}\int_{A_i} \left(\begin{pmatrix} w_{i2}\dot{w}_{i3} - w_{i3}\dot{w}_{i2} \\ w_{i3}\dot{w}_{i1} - w_{i1}\dot{w}_{i3} \\ w_{i1}\dot{w}_{i2} - w_{i2}\dot{w}_{i1} \end{pmatrix} + \begin{pmatrix} r_{i2}\dot{w}_{i3} - r_{i3}\dot{w}_{i2} \\ r_{i3}\dot{w}_{i1} - r_{i1}\dot{w}_{i3} \\ r_{i1}\dot{w}_{i2} - r_{i2}\dot{w}_{i1} \end{pmatrix} \right.$$

$$+\left\{\begin{bmatrix} (w_{i3}+r_{i3})^2+(w_{i2}+r_{i2})^2 & -(w_{i2}+r_{i2})(w_{i1}+r_{i1}) & -(w_{i1}+r_{i1})(w_{i3}+r_{i3}) \\ -(w_{i1}+r_{i1})(w_{i2}+r_{i2}) & (w_{i3}+r_{i3})^2+(w_{i1}+r_{i1})^2 & -(w_{i3}+r_{i3})(w_{i2}+r_{i2}) \\ -(w_{i1}+r_{i1})(w_{i3}+r_{i3}) & -(w_{i2}+r_{i2})(w_{i3}+r_{i3}) & (w_{i2}+r_{i2})^2+(w_{i1}+r_{i1})^2 \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\right]dm_i\Bigg\} \tag{A.16}$$

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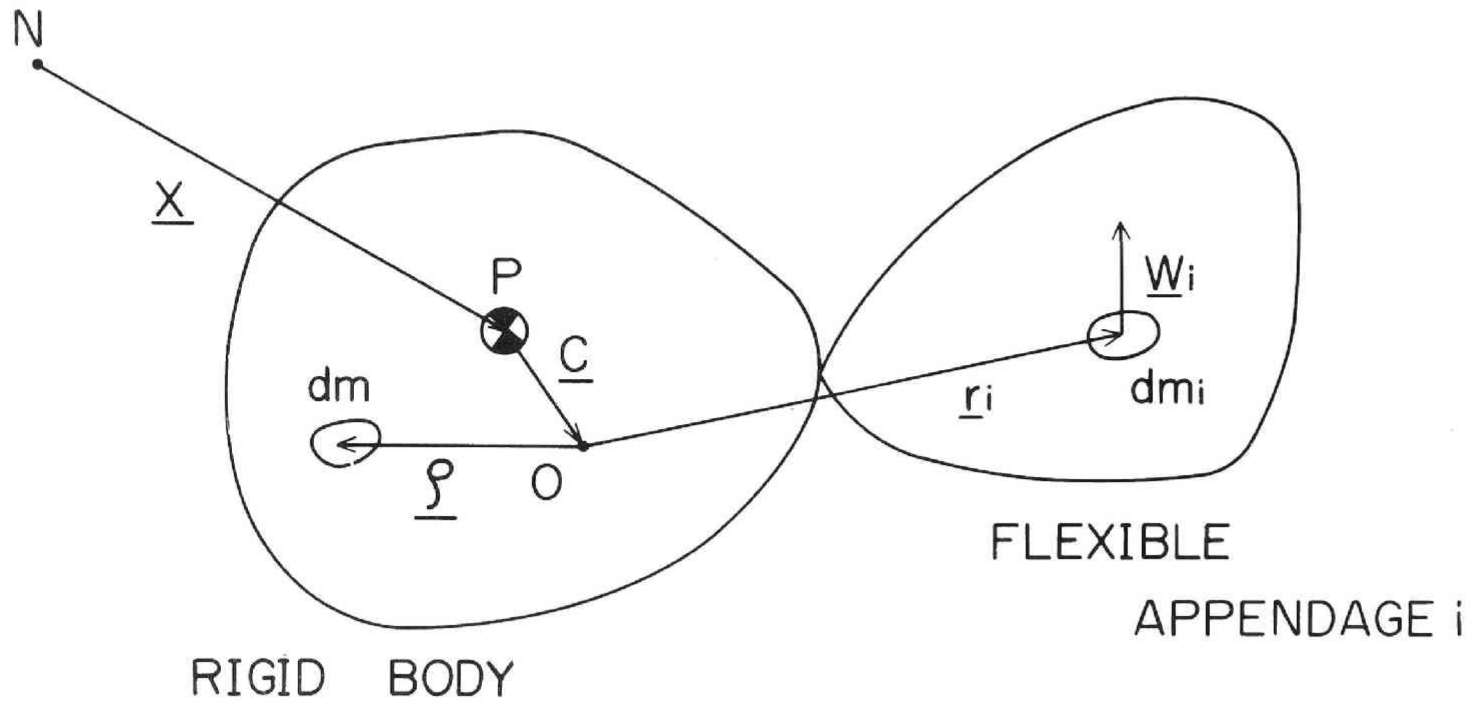

Fig. A.1 Flexible spacecraft configuration